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Speakers: J. M. Anderson (London), P. M. Gauthier (Montreal), B. Goldsmith (DIT), A. J. O'Farrell (Maynooth), J. V. Pulé (UCD), R. Ryan (UCG).

Requests for accommodation should be submitted by 1 July, 1994.
Conference dinner on Monday 5 September, 1994.
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Polynomials and Holomorphic Functions on Infinite Dimensional Spaces

7–9 September, 1994

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TRACE-ZERO MATRICES AND POLYNOMIAL COMMUTATORS

T. J. Laffey and T. T. West

Let $\mathbf{F}$ denote a field and $M_n(\mathbf{F})$ the algebra of $n \times n$ matrices over the field $\mathbf{F}$. If $X \in M_n(\mathbf{F})$, $\mathrm{tr}(X)$ will denote the trace of the matrix X. A well known result of Albert and Muckenhoupt [1] states that if $\mathrm{tr}(X) = 0$ then there exist matrices A, $B \in M_n(\mathbf{F})$ such that X is the commutator of A and B,

$$X = [A, B] = AB - BA.$$

Let p denote a polynomial in $\mathbf{F}[x]$ of degree greater than or equal to one. The *Polynomial Commutator* of A and B relative to p is defined to be

$$p[A, B] = p(AB) - p(BA).$$

It is easy to check, by examining the eigenvalues, that $\mathrm{tr}(p[A, B])$ is always zero. The Albert-Muckenhoupt result states that if $X \in M_n(\mathbf{F})$ with $\mathrm{tr}(X) = 0$ then, for $p(x) = x$,

$$X = p[A, B],$$

for some $A, B \in M_n(\mathbf{F})$. We show that, if the field $\mathbf{F}$ has characteristic zero the Albert-Muckenhoupt result may be extended to general polynomials of degree greater than, or equal to, one.

Theorem. *Let $\mathbf{F}$ be a field of characteristic zero and let $p \in \mathbf{F}[x]$ have degree greater than or equal to one. If $X \in M_n(\mathbf{F})$ is of trace zero then there exist matrices A, $B \in M_n(\mathbf{F})$ such that*

$$X = p[A, B].$$

First we prove the following elementary

Lemma. *If $\mathbb{F}$ is a field of characteristic zero and $X \in M_n(\mathbb{F})$ is of trace zero then we can choose a basis of $\mathbb{F}^n$ such that, relative to this basis, X has zeros on its main diagonal.*

Proof: Since $\operatorname{tr}(X) = 0$ and $\mathbb{F}$ is of characteristic zero, X is not a scalar matrix. Thus there exists a vector $v \in \mathbb{F}^n$ such that v and Xv are linearly independent.

Set $v_1 = v$, $v_2 = Xv$ and extend to a basis $v_1, v_2, \ldots, v_n$ of $\mathbb{F}^n$. Relative to this basis

$$X = [x_{ij}]_{n\times n} \qquad \text{with } x_{11} = 0.$$

Further the matrix

$$Y = [x_{ij}]_{(n-1)\times(n-1)} \qquad (2 \le i, j \le n)$$

has trace zero and the proof may be completed by induction.

Proof of Theorem: Since $\operatorname{tr}(X) = 0$ we may take

$$X = [x_{ij}]_{n\times n} \quad \text{with } x_{ii} = 0 \qquad (1 \le i \le n).$$

Now

$$X = L - U,$$

where L is a lower triangular matrix, U is an upper triangular matrix and both have zeros on the main diagonal.

Let D be the diagonal matrix

$$D = \operatorname{diag}(d_1, \ldots, d_n),$$

then $p(D)$ is the diagonal matrix

$$p(D) = \operatorname{diag}(p(d_1), \ldots, p(d_n)),$$

and since $\mathbb{F}$ is an infinite field and the degree of p is greater than, or equal to, one, we may choose the d_i so that the $p(d_i)$ are distinct $(1 \le i \le n)$.

Then

$$\begin{aligned} X &= (L + p(D)) - (U + p(D)) \\ &= L_1 - U_1 \end{aligned}$$

where $L_1 = L + p(D)$ is lower triangular and $U_1 = U + p(D)$ is upper triangular. The diagonal entries of L_1 and U_1 are $p(d_i)$, $(1 \le i \le n)$, and since these have been chosen distinct, the matrices L_1, U_1 and $p(D)$ are all similar. Thus there exist invertible $S, T \in M_n(\mathbb{F})$ so that

$$\begin{aligned} X &= S^{-1}p(D)S - T^{-1}p(D)T, \\ &= p(S^{-1}DS) - p(T^{-1}DT). \end{aligned}$$

Taking $A = S^{-1}T$ and $B = T^{-1}DS$ gives

$$X = p(AB) - p(BA) = p[A, B] \qquad (*)$$

which completes the proof.

Remarks

1. The result does not remain true if the restriction that $\mathbb{F}$ is of characteristic zero be dropped.
2. It would be interesting to investigate the latitude in equation $(*)$, for fixed X and p, in the possible choices of A and B.

Reference

[1] A. A. Albert and B. Muckenhoupt, *On matrices of trace zero*, Michigan J. Math. 4 (1957), 1–3.

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