HOW TO COMPOSE A PROBLEM FOR THE INTERNATIONAL MATHEMATICAL OLYMPIAD?

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A difficult task for the organizers of any national team for participation in the International Mathematical Olympiad (IMO) is to fulfill the request of the host nation to submit original problems for consideration by the jury for inclusion in the Olympiad. To compose such problems requires considerable skill and even mathematicians of a high calibre can find the task difficult because many of the techniques of the professional mathematician are excluded by the requirement that the problems have "elementary" solutions. Arthur Engel has written a very interesting article [2] describing his thought processes in composing IMO problems. Since the author is actively involved in the preparation and training of the Irish IMO team he has felt it incumbent on himself to compose suitable problems. This article describes some of his attempts.

In this article four avenues of approach to the task of composing IMO problems are considered. They are:
§1. Use a known result in some area of mathematics that might reasonably be assumed to be outside the knowledge of the contestants.
§2. Do a variation on a known elementary, but tricky, result.
§3. Compose a problem from a topic being currently taught by the composer.
§4. Use someone else's problem!

§1. In past Olympiads some of the problems which appeared were direct applications of a piece of mathematics which is well-known

to mathematicians, e.g. the pigeon-hole principle or the concept of an eigenvalue. But nowadays it is taken for granted that such problems would be deemed too trivial, because of the training that many of the contestants receive. The first idea the author had for constructing an Olympiad problem was to take a known result in some area of mathematics that might be reasonably assumed to be outside the knowledge of the contestants and to vary it a little. Thus, on page 3 of Jacobson's book [3] on Jordan algebras is the following result of Hua Loo Keng:

Theorem. *Let σ be an additive mapping of a division ring Δ into a division ring Δ' which preserves inverses. Then σ is either a homomorphism or an antihomomorphism.*

The question we ask is: does this give a non-trivial problem for the real numbers? As an answer we have

Problem 1. Let f be a function from the real numbers to the real numbers such that $f(1) = 1$, $f(a+b) = f(a) + f(b)$ for all a and b and $f(a)f(1/a) = 1$ for all $a \neq 0$. Prove that $f(x) = x$ for all x.

Proof. It is easy to prove that the properties $f(1) = 1$ and $f(a+b) = f(a) + f(b)$ for all a and b imply $f(x) = x$ for all rational numbers x. It is also not difficult to prove that f is injective and that $f(-x) = -f(x)$ for all x.
Next we note that, if $f(a) \neq f(a^2)$,

$$\begin{aligned} 1/[f(a) - f(a^2)] &= 1/\big[f(a(1-a))\big] \\ &= f[1/a + 1/(1-a)] \\ &= 1/[f(a) - f(a)^2]. \end{aligned}$$

Thus $f(a^2) = f(a)^2$ here and this result is still true when $f(a) = f(a^2)$. Thus $f(x) > 0$ if $x > 0$. So $a > b$ implies $f(a) > f(b)$.

Finally, if x is any real number there exist two sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers such that x is the only real number satisfying the condition $a_n < x < b_n$ for all natural numbers n. Then $a_n = f(a_n) < f(x) < f(b_n) = b_n$ holds for all n and hence $f(x) = x$ for all real numbers x.

This problem was submitted to the 30th IMO in Germany in 1989 but was not shortlisted. Perhaps it was not suitable because a characterization of the real numbers might not be known to some contestants.

Another, more recent, result that it was felt might yield a suitable problem is the following result of Leep and Shapiro [5]:

Theorem. *Let G be a subgroup of index 3 of the multiplicative group of a field F. Then every element of F is expressible in the form $g+h$ where g and h are elements of G, except when $|F|=4$, 7, 13 or 16.*

Replacing F by the rational numbers doesn't seem to make the theorem any easier and, in any case, one shouldn't expect too many of the IMO contestants to know much about groups. But the theorem is the motivating idea for the following problem.

Problem 2. Let $\mathbf{Q}$ denote the set of rational numbers. Let S be a nonempty subset of $\mathbf{Q}$ with the properties:

(i) $0 \notin S$;

(ii) if $s_1, s_2 \in S$ then $s_1/s_2 \in S$;

(iii) there exists $q \in \mathbf{Q}$ with $q \neq 0$ such that every nonzero rational number not in S is of the form qs for some $s \in S$.

Prove that if $x \in S$ then there exist $y, z \in S$ such that $x = y + z$.

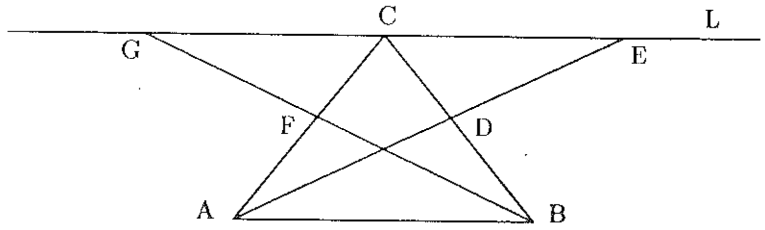
This problem is too easy for an IMO but it was included in the 1991 Irish Mathematical Olympiad and gave a lot of difficulty to the contestants because of the group theory concepts involved.

§2. Another idea for composing a problem is to take a known elementary, but tricky, result and do a variation of it. For example, the following is a well-known, difficult result in plane geometry.

The Steiner-Lehmus Theorem. *[1]. If the bisectors of two angles of a triangle are equal in length then the triangle is isosceles.*

The following variation suggested itself.

Problem 3. Let ABC be a triangle with L a line through C parallel to the side AB. Let the internal bisector of the angle at A meet BC at D and L at E, and let the internal bisector of the angle at B meet AC at F and L at G. If $DE = FG$ prove that $CA = CB$.



Proof. Let $BC = a$, $CA = b$, $AB = c$ and $A = 2\alpha$, $B = 2\beta$. Obviously

$$\frac{CF}{FA} = \frac{a}{c}, \qquad CF = \frac{ab}{a+c}, \qquad CD = \frac{bc}{b+c}.$$

Suppose $a > b$. Then $\alpha > \beta$, $\sin\alpha > \sin\beta$ and $\sin 2\alpha > \sin 2\beta$. In the triangle CFG

$$\frac{GF}{\sin 2\alpha} = \frac{CF}{\sin\beta}$$

and thus

$$GF = \frac{ab\sin 2\alpha}{(a+c)\sin\beta}.$$

In the triangle CDE

$$\frac{CD}{\sin\alpha} = \frac{ED}{\sin 2\beta}$$

and thus

$$ED = \frac{ab\sin 2\beta}{(b+c)\sin\alpha}.$$

Since $GF = ED$ we get

$$(b+c)\sin\alpha\sin 2\alpha = (a+c)\sin\beta\sin 2\beta.$$

Using the fact that $\dfrac{a}{\sin 2\alpha} = \dfrac{b}{\sin 2\beta}$ we get

$$\frac{\sin\alpha}{\sin\beta} = \frac{(a+c)b}{(b+c)a} = \frac{ab+bc}{ab+ac} < 1$$

since $a > b$. This implies $\sin\alpha < \sin\beta$, which contradicts the assumption $a > b$. It follows that $a \le b$. In the same way, $a \ge b$. Thus $a = b$.

Proofs of this result using Euclidean geometry and coordinate geometry have also been found.

This problem made it to the short list at the 31st IMO in Beijing in 1990 but did not feature in the final Jury discussions.

§3. Another area of inspiration for composing IMO problems is whatever the composer is teaching at the time! Thus, in teaching a course on complex analysis the author felt that the topic of Möbius transformations should yield a tricky problem. And, sure enough, we have

Problem 4. Let P be the set of positive rational numbers and let the function f from P to itself have the properties

(i) $f(x) + f(1/x) = 1$ and

(ii) $f(2x) = 2f(f(x))$ for all $x \in P$.

Determine, with proof, a formula for $f(x)$.

Proof. Let $x = 1$ in (i) to get $f(1) = \dfrac{1}{2}$.

Then (ii) yields $f(2) = 2f\left(\dfrac{1}{2}\right)$ and, putting $x = 2$, (i) gives $f(\frac{1}{2}) = \dfrac{1}{3}$ and $f(2) = \dfrac{2}{3}$. Trying a few more values of x leads one to suspect that $f(x) = \dfrac{x}{x+1}$ for all $x \in P$. If $x \in P$ then x can be written as $\dfrac{m}{n}$ where m and n are relatively prime natural numbers and we shall assume that all the rational numbers we deal with are expressed in this reduced form. Let $h(x) = m + n$ where $x = \dfrac{m}{n}$ is in reduced form. We prove that $f(x) = \dfrac{x}{x+1}$ by induction on $h(x)$.

It is clear that $\dfrac{x}{x+1}$ satisfies properties (i) and (ii) of the problem.

Next, $h(x) = 2$ forces $x = 1$ and $h(x) = 3$ forces $x = \dfrac{1}{2}$ or 2. Thus we have already verified the formula for $f(x)$ when $h(x) \le 3$. So let $x \in P$ with $h(x) > 3$ and assume the formula for $f(y)$ holds for all $y \in P$ with $h(y) < h(x)$.

Let $x = \dfrac{m}{n}$ be in reduced form and suppose m and n are both *odd*. Suppose also, without loss of generality, that $m < n$. Since

$$h\left(\frac{m}{n-m}\right) = n < m + n = h(x)$$

we have

$$f\left(\frac{m}{n-m}\right) = \frac{m}{n}.$$

There exists a natural number d such that $n - m = 2d$. Thus

$$\begin{aligned} f(x) &= f\left(f\left(\frac{m}{n-m}\right)\right) \\ &= \frac{1}{2}f\left(\frac{2m}{n-m}\right) \\ &= \frac{1}{2}f\left(\frac{m}{d}\right) \\ &= \frac{m}{2(m+d)} \end{aligned}$$

by the induction hypothesis, since

$$\begin{aligned} h\left(\frac{m}{d}\right) &= m + \frac{1}{2}(n-m) \\ &= \frac{m+n}{2} < h(x). \end{aligned}$$

Hence

$$f(x) = \frac{m}{m+n} = \frac{x}{x+1}.$$

Thus the formula holds if m and n are both odd.

Now suppose one of m and n, m say, is even. Then $m = 2^k r$ for some integer $k \geq 1$ and some odd integer r. Then we get

$$\begin{aligned} f(x) &= f\left(\frac{2^k r}{n}\right) \\ &= 2f\left(f\left(\frac{2^{k-1}r}{n}\right)\right) \\ &= 2f\left(\frac{2^{k-1}r}{2^{k-1}r+n}\right) \\ &= \\ &\vdots \quad \vdots \\ &= \\ &2^k f\left(\frac{r}{(2^k-1)r+n}\right) \end{aligned}$$

and we note that

$$h\left(\frac{r}{(2^k-1)r+n}\right) = h(x).$$

Letting $m_1 = m$ and $n_1 = n$ we have proved that there exist natural numbers m_2 and n_2, with m_2 even and n_2 odd, so that $f(m_1/n_1) = 2^k f(n_2/m_2)$ and $h(m_1/n_1) = h(m_2/n_2)$. We then have $f(m_1/n_1) = 2^k[1 - f(m_2/n_2)]$. Repeating this process we get a sequence of positive rationals m_i/n_i with m_i even and n_i odd, $f(m_i/n_i) = 2^{k_i}[1 - f(m_{i+1}/n_{i+1})]$, where k_i is the highest power of 2 dividing m_i and $h(m_i/n_i) = h(x)$ for $i = 1, 2, \ldots$

Since there are only finitely many rationals satisfying the last condition there exist natural numbers r and s with $r < s$ so that $m_r/n_r = m_s/n_s$. Then there exist integers p and q so that $f(m_r/n_r) = p + qf(m_s/n_s)$ and $q = \pm 2^t$ for some natural number t. Hence we get the unique value $p/(1-q)$ for $f(m_r/n_r)$. Thus we get a unique value for $f(x)$. Since only properties (i) and (ii) were used in deriving this value of $f(x)$, and $\frac{x}{x+1}$ satisfies these properties, uniqueness implies that $f(x) = \frac{x}{x+1}$. So the result is true by induction.

It is probably clear how this problem was composed: write down a few properties of the function $\frac{x}{x+1}$ and try to recover the original function from these properties.

This problem was also included in the 1991 Irish Mathematical Olympiad. It was intended originally to submit the problem for consideration by the IMO jury in Sweden in 1991 but, owing to an error in the author's original solution, the problem was considered too easy. T. J. Laffey supplied the crucial argument for dealing with $\frac{m}{n}$ when m is even and n is odd.

It can happen with IMO-type problems that a very elegant solution can be produced which the composer did not envisage (The author would welcome such a solution to Problem 4 above!). At the 30th IMO in Braunschweig the following problem was accompanied by a very complicated solution and was rated A++ by the jury:

Problem 5. A permutation $(x_1, x_2, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

My colleague Mícheál Ó Searcóid came up with this elegant proof of a more general result:

We say that the permutation $(x_1, x_2, \ldots, x_{2n})$ has an an *adjacent pair* if and only if $|x_i - x_{i+1}| = n$ for some i with $1 \leq i \leq 2n-1$. Let S be the set of those permutations with *exactly one* adjacent pair and let T be the set of permutations with no adjacent pairs. We shall prove that $|S| > |T|$. This is clear if $n = 1$. So let $n > 1$. Let $f : T \to S$ such that $f(x_1, \ldots x_{2n}) = (x_2, x_3, \ldots, x_{j-1}, x_1, x_j, \ldots, x_{2n})$ where $|x_1 - x_j| = n$. Then f is well-defined and injective. It is not surjective since, for example, the permutation $(1, n+1, 2, 3, \ldots n, n+2, \ldots 2n)$ is in S but not in T. Hence $|S| > |T|$. This more general result now implies the proof of problem 5.

§4. One final way of getting a problem for the IMO is to use someone else's! The other "Irish" problem shortlisted for the IMO in Beijing is the creation of Charles Johnson of the College of William and Mary in Virginia. It is:

Problem 6. An eccentric mathematician has a ladder with n rungs which he ascends and descends in the following way: whenever he ascends each step he takes covers a rungs of the ladder and whenever he descends each step he takes covers b rungs of the ladder. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and climb down to ground level again. Find, with proof, the smallest value of n, expressed in terms of a and b.

Solution. The smallest value of n is $a + b - (a, b)$ where (a, b) is the greatest common divisor of a and b. This is obvious if $a|b$ or $b|a$.

Suppose that $(a, b) = 1$. Suppose also that $a > b$ (there is no loss of generality since the problem is symmetric with respect to ascending or descending the ladder). Then there exist natural numbers r_1, s_1 so that

$$a = bs_1 + r_1$$

where $0 < r_1 < b$. In general, given the remainder r_{j-1}, there exist integers r_j and s_j so that $a + r_{j-1} = bs_j + r_j$, where $0 \leq r_j \leq b-1$, for $j = 2, 3, \ldots$. Since $a \equiv r_1 \pmod b$ we get $r_j \equiv jr_1 \pmod b$, for $j = 1, 2, \ldots$. Since $(r_1, b) = 1$ the integers $r_1, r_2, \ldots, r_b$ are distinct and thus are equal to $0, 1, 2, \ldots, b-1$ in some order. We must have $r_b = 0$, since $r_j = 0$, for some $j < b$, implies that $r_{j+1} = r_1$, which is a contradiction. If the mathematician is standing on rung r_j, (counted from the bottom) of the ladder and $a + r_j \leq n$ then, by ascending by a rungs and descending by bs_{j+1} rungs, he can get to rung r_{j+1}. So, if $a + b - 1 = n$, we have $a + r_j \leq n$ for $j = 1, 2, \ldots, b$ and, since $b - 1 = r_j$ for some j, he can clearly get to rung r_j for each $j = 1, 2, \ldots, b$ and thus he can climb to the top rung of the ladder and back to ground level again. If $n < a + b - 1$ he can not reach rung r_j for some $j \leq b$ and thus he can not reach "rung" r_b, i.e. ground level, and thus he can not ascend and descend the ladder in the required way. So the smallest value of n is $a+b-1$. Finally, if $(a, b) = k > 1$ replace a and b in the above discussion by a/k and b/k, respectively, and then scale all integers up by k to get $a + b - (a, b)$ as the smallest value of n.

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