In summary I would recommend that most mathematicians should have this book on their shelves. Any minor faults with the book are due to limitations of space, but I do feel that it was a pity that not even a brief discussion of numerical solution of differential equations was not included, although I am aware that this omission was probably deliberate.

P. O'Leary
Dept. of Mathematical Physics
U.C.G., Galway

RIA PROCEEDINGS

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PROBLEM PAGE

Editor: Phil Rippon

First, here is a very pretty problem, which I heard about from Tom Laffey. It appeared in the International Mathematical Olympiad 1986 at Warsaw, and was the hardest problem set, in terms of the total scores of all candidates on individual questions. Nevertheless, several candidates solved the problem and an American student, Joseph Keane was awarded a special gold medal for his solution.

1. *To each vertex of a regular pentagon, an integer is assigned in such a way that the sum of all five integers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$, then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five integers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.*

Next, a problem from John Mason at the Open University, who says that it is known in Maths. Education circles as the Krutetskii Problem. I have also seen it attributed to Lovacz.

2. *A finite number of petrol dumps are arranged around a racetrack. The dumps are not necessarily equally spaced and nor do they necessarily contain equal volumes of petrol. However, the total volume of petrol is sufficient for a car to make one circuit of the track. Show that the car can be placed, with an empty tank, at some dump so that, by picking up petrol as it goes, it can complete one full circuit.*

John Mason also asked the following apparently much harder problem. I am not aware of any reference to this problem in the literature.

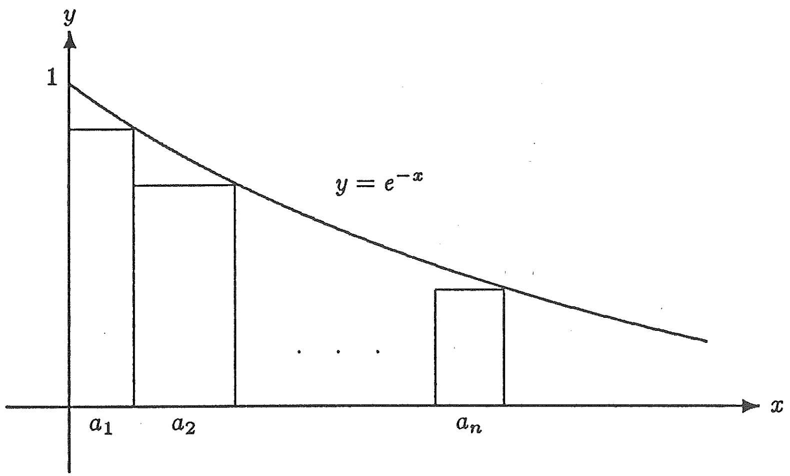
3. *The petrol dumps are arranged as in Problem 2, but this time the total volume of petrol is sufficient for two circuits of the track. Can two cars be placed, with empty tanks, at the same dump so that, by picking up petrol as they go, they can each complete one full circuit in* opposite *directions? (The cars may cooperate in sharing petrol from the dumps.)*

Now, to earlier problems. Finbarr Holland has sent an alternative solution to Tom Carroll's problem:

If $a_n \geq 0$, for $n = 1, 2, \cdots$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{e^{a_1+a_2+\cdots+a_n}} < 1$$

Finbarr's proof can be expressed most succinctly using the following picture:



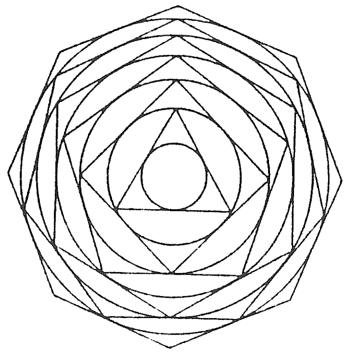
The picture clearly demonstrates that

$$\sum_{n=1}^{\infty} a_n e^{-(a_1+a_2+\cdots+a_n)} < \int_0^{\infty} e^{-x}\, dx = 1.$$

This proof was shown to me recently by Aimo Hinkkanen also.

Next, here are solutions to the problems which appeared in December 1986.

1. The radii of the circles in the following expanding pattern (in which the radius of the innermost circle is 1) tend to a limit which is approximately 8.7.



If at each stage we *double* the number of sides of the escribed polygons, then the limiting radius can be found explicitly. What is it?

In the above diagram the radius of the nth outer circle is

$$[\cos(\pi/3)\cos(\pi/4)\cdots\cos(\pi/(n+2))]^{-1}$$

Since $\cos x \geq 1 - x^2/2$, the infinite product

$$\prod_{n=3}^{\infty} \cos(\pi/n) \geq \prod_{n=3}^{\infty} (1 - \pi^2/2n^2) > 0$$

because

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty\,.$$

If at each stage we *double* the number of sides of the escribed polygons, then the radius of the nth outer circle is

$$\left[\cos(\pi/3)\cos(\pi/6)\cdots\cos(\pi/(3.2^{n-1}))\right]^{-1}.$$

This product can in fact be simplified by a trick due to Euler, based on the half-angle formula:

$$\cos(\theta/2) = \frac{\sin\theta}{2\sin(\theta/2)}, \qquad 0 < \theta < 2\pi.$$

Repeated use of this formula gives

$$\begin{aligned}\cos(\theta/2)\cos(\theta/4)\cdots\cos(\theta/2^n) &= \frac{\sin\theta}{2\sin(\theta/2)}\frac{\sin(\theta/2)}{2\sin(\theta/4)}\cdots\frac{\sin(\theta/2^{n-1})}{2\sin(\theta/2^n)}\\ &= \frac{\sin\theta}{2^n\sin(\theta/2^n)},\end{aligned}$$

Since most terms cancel. If we now use the fact that

$$\lim_{n\to\infty}\frac{\sin(\theta/2^n)}{\theta/2^n} = 1$$

then we obtain

$$\prod_{n=1}^{\infty}\cos(\theta/2^n) = \frac{\sin\theta}{\theta}$$

On substituting $\theta = 2\pi/3$, we get

$$\prod_{n-1}^{\infty}\cos(\pi/(3.2^{n-1}) = \frac{\sin(2\pi/3)}{2\pi/3} = \frac{3\sqrt{3}}{4\pi}.$$

Hence the limiting outerradius in this problem is $4\pi/3\sqrt{3} \simeq 2.42$.

2. If $f(x) = p(x)e^x$, where p is a quadratic with integer coefficients, is it possible for f, f' and f'' to have rational zeros?

This problem was told me by John Reade at Manchester. Unfortunately, for setters (and solvers!) of curve-sketching problems, the answer is 'no'. Here is John's solution to the problem.

We may assume that

$$p(x) = x^2 + \lambda x + \mu, \qquad \lambda,\ \mu \text{ rational}.$$

The conditional for p to have rational zeros is

$$\lambda^2 - 4\mu = s^2, \qquad s \text{ rational}.$$

Now

$$f'(x) = (p(x) + p'(x))\,e^x = \left(x^2 + (\lambda+2)x + \mu + \lambda\right)e^x,$$

and so the condition for f' to have rational zeros is

$$(\lambda+2)^2 - 4(\mu+\lambda) = t^2, \qquad t \text{ rational},$$

which reduces to

$$s^2 + 4 = t^2. \tag{1}$$

Similarly, the condition for f'' to have rational zeros reduces to

$$t^2 + 4 = u^2, \qquad u \text{ rational}. \tag{2}$$

Multiplying (1) and (2) by the common denominator of s^2, t^2, u^2, we obtain

$$\begin{cases} a^2 + d^2 = b^2 \\ b^2 + d^2 = c^2 \end{cases} \qquad a,\ b,\ c,\ d \text{ integers} \tag{3}$$

Now we use the method of infinite descent to show that thre are *no solutions* to (3). The idea is to show that if a solution a, b, c, d to (3) exists, then it is possible to construct a smaller solution α, β, γ, δ to (3). Repeated application of this argument leads to a contradiction, and so (3) has no solutions.

We need a well-known result on Pythagorean triples (see almost any book on number theory).

Lemma *If $x^2 + y^2 = z^2$, where x, y, z are comprime, then z is odd and exactly one of x, y is even. If x is even, then*

$$x = 2uv, \quad y = u^2 - v^2, \quad z + u^2 + v^2,$$

for some comprime integers u, v.

If the equations (3) have a solution a, b, c, d, then we may assume that a, b, c, d have no common factors, and hence that each pair is comprime. The lemma now implies that b, c are odd, so that d is even and a is odd. Hence

$$h = \frac{1}{2}(a+c), \qquad k = \frac{1}{2}(c-a)$$

are positive integers, and it is easy to check that

$$h^2 + k^2 = b^2, \qquad 2hk = d^2 .$$

Also h, k are comprime with each other and with b.

Now suppose that h is even (the argument is similar if k is even). Then, by the lemma,

$$h = 2uv\,, \quad k = u^2 - v^2\,, \quad b = u^2 + v^2\,,$$

where u, v are comprime. Thus

$$d^2 = 2hk = 4uv(u-v)(u+v)\,,$$

where u, v, $u-v$, $u+v$ are pairwise comprime. Hence each of u, v, $u-v$, $u+v$ is a perfect square, say

$$u = \beta^2\,, \quad v = \delta^2\,, \quad u - v = \alpha^2\,, u + v = \gamma^2 .$$

Since this gives

$$\begin{aligned} \alpha^2 + \delta^2 &= \beta^2 \\ \beta^2 + \delta^2 &= \gamma^2 \end{aligned}$$

and $\beta^4 = u^2 < b$, the proof that (3) has no solutions is complete.

3. Which integers can be expressed as the sum of two or more consecutive positive integers?

A quick check of the integers 1, 2, 3, $\cdots$, 10, say, suggests the conjecture that all positive integers except the sequence 1, 2, 4, 8, $\cdots$ can be expressed in this way: $3 = 1+2$, $5 = 2+3$, $6 = 1+2+3$, $7 = 3+4$, $9 = 4+5$, $10 = 1+2+3+4$.

This conjecture turns out to be true and, surprisingly, there is an 'easy' proof.

First we check that any sum of consecutive integers must have an odd factor (> 1). Here is one way to verify this:

The sum of an odd number $2m+1$ of consecutive integers can be written in the form:

$$\underbrace{(n-m) + \cdots + \underbrace{(n-1) + n + (n+1)}_{\vdots} + \cdots + (n+m)} = (2m+1)n\,; \qquad (4)$$

the sum of an even number $2n$ of consecutive integers can be written in the form:

$$\underbrace{(m-n+1) + \cdots + \underbrace{m + (m+1)}_{\vdots} + \cdots + (m+n)} = (2m+1)n\,. \qquad (5)$$

This proves that the integers $1, 2, 4, 8, \cdots$ cannot be expressed as sum of two or more consecutive integers.

At first sight, it seems much harder to prove that all other positive integers *can* be expressed in this way. However, any other positive integer can be written in the form $(2m+1)n$, where m, n are positive integers. Moreover, we have either $n - m \geq 1$ or $m - n + 1 \geq 1$. Hence $(2m+1)n$ can be expressed as the sum of consecutive positive integers using either (4) or (5).

Phil Rippon
Faculty of Mathematics
Open University
Milton Keynes, MK7 6AA, UK