Integrating Inverse Functions

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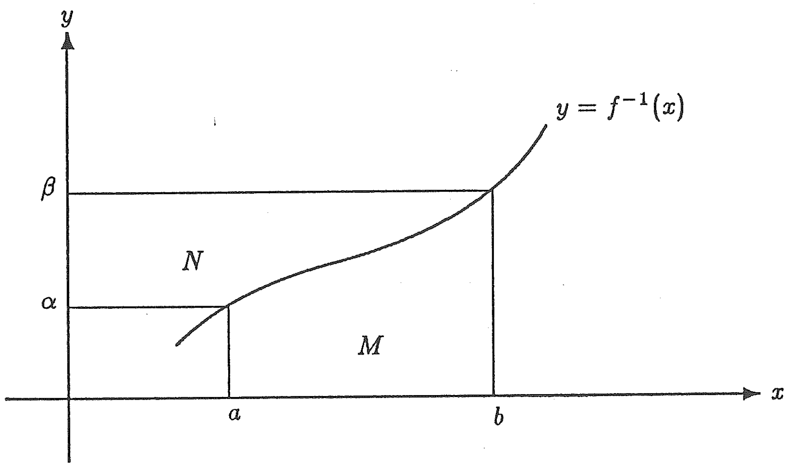
The formula for the derivative of an inverse function is given in every calculus textbook, but is rarely, if ever, pointed out that there is also a formula for the integral of an inverse function. The formula is:

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad (1)$$

where $y = f^{-1}(x)$ or for definite integrals

$$\int_a^b f^{-1}(x)\,dx = [xf^{-1}(x)]_a^b - \int_\alpha^\beta f(y)\,dy \qquad (2)$$

where $\alpha = f^{-1}(a)$ and $\beta = f^{-1}(b)$. The derivation of this formula is an easy application of integration by parts, taking $u = f^{-1}(x)$ and $v = x$.



The figure gives a graphical interpretation of (2) in the case where f is increasing. The definite integrals give the areas M and N and the term $[xf^{-1}(x)]_a^b$ expresses $M + N$ as the difference of two rectangles. A difference picture is needed for decreasing functions — the details are left to the reader.

With this formula the integrals of many of the standard inverse functions can be computed directly, without working through the details of integration by parts in each case as is usually done. For example,

$$\begin{aligned}\int \arcsin x\,dx &= x\arcsin x - \int \sin y\,dy \\ &= x\arcsin x + \cos y \\ &= = x\arcsin x + \sqrt{1-x^2}.\end{aligned}$$

Thus, if the integral of f is known, we can immediately write down the integral of f^{-1}.

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