[17] Pieri, Mario, *La geometria elementare instituita sulle nazioni di "punto" e "sfera"*. Memorie di Matematica e di Fisica della Societa Italiana delle Scienze, Ser. 3, 15, 1908

College of Technology
Mathematics Department
Kevin Street, Dublin 8

Reciprocity Agreements

IMTA Reciprocity Membership

Associate membership of the Irish Mathematics Teachers Association is available to members of the Irish Mathematical Society at a specially reduced rate. Applications should be made through the IMS Treasurer

AMS Reciprocity Membership

Membership of the American Mathematical Society is available to members of the Irish Mathematical Society at the reduced rate of $42. Application forms can be obtained from the Secretary

MATHEMATICAL EDUCATION

Approaches To School Geometry

Mary Morrin David Murphy Ann O'Shea

Introduction

This article arises from a postgraduate course in geometry given by Professor Barry at U.C.C. As part of the course we undertook some project work on the geometry courses of Georges Papy, Gustave Choquet and Jean Dieudonne. Here we hope to review these three courses and their potential for inclusion in the secondary school curriculum.

First of all, we must ask the question: why teach geometry? One obvious reason for teaching geometry is its application to real life situations and problems. Through the study of geometry children develop practical skills in such areas as measurement, calculations of areas and volumes, use of grids and co-ordinate systems. It also gives them an understanding of the concepts of two-dimensional and three-dimensional space. Clearly geometry has application to topics in mathematics and can indeed be regarded as a unifying theme in the mathematics curriculum. It provides a rich source of visualisation for arithmetical and algebraic concepts. Geometry is essential for mastering calculus and therefore all other fields that have calculus as a prerequisite. A major reason for the inclusion of geometry in the secondary school curriculum is its value as a vehicle for stimulating and exercising general thinking skills, skill in deductive reasoning and problem solving. Through its precise use of language, geometry can also play a part in the development of skills in communication. Therefore, geometry has an important role in the secondary school curriculum.

The next question is: How should we teach geometry in secondary schools? It seems to us that there are two main approaches. One the one hand there is the synthetic approach, which was used by Euclid and later completed and brought to logical perfection by he German mathematician David Hilbert. On the other hand we have an approach to geometry which uses linear algebra. Choquet, Papy and Dieudonne all favour the latter.

We now outline their courses.

Choquet

While Choquet agrees that children benefit from an approach to geometry based on concepts drawn from the real world such as parallelism, perpendicularity and distance he believes that from the mathematician's point of view the most valuable method of defining a plane as a 2-dimensional vector space over $\mathbb{R}$ having an inner product. In order to reconcile these ideas he uses synthetic axioms and sets out to demonstrate the algebraic structure of the plane. Then using the tools of linear algebra he develops the course in geometry.

Choquet's first step is to develop the vector space structure of the plane. His initial axioms are concerned with incidence properties of points and lines and also deal with parallelism. Parallel projection is the natural orders on a line and so he can deal with betweenness of points. His next axiom assumes that parallel projection maps intervals to intervals and therefore, preserves betweeness of points. Choquet makes a strong point that geometry should not be burdened with the task of constructing the real numbers and in his courses he assumes that $\mathbb{R}$ is a totally ordered archimedean commutative field. His next axiom assumes distance on a line.

Now Choquet has both distance and order on every line D, so he can chose any point $o \in D$ and one of the natural orders of D to obtain a pair (D, o) called a pointed line. On this pointed line he can define operations under a unique isomorphism. Thus each line (D, o) is a vector space. Choquet next defines midpoints and postulates that parallel projection preserves midpoints. He can now define a parallelogram as a quadruplet of points (a, b, a', b') such that (a, a') and (b, b') have the same midpoint. Having chosen any point $o \in D$ as origin and writing (Π, o) for the plane Π with origin chosen at o. Choquet defines addition in Π to be the operator $(x, y) \to x + y$ where $(0, x, x + y, y)$ is a parallelogram. He can show that addition is well defined and proves that $((\Pi, o), +)$ is an abelian group. He next defines scalar multiplication and shows that $(\Pi, 0)$ is a vector space. He uses translations and homothetic maps to show that for any $a, b \in \Pi$ the vector spaces (Π, a) and (Π, b) are isomorphic. Having established the vector (space) structure of the plane Choquet discusses some affine transformations.

The next step is to obtain distance on the plane. Here two further axioms are introduced. The first introduces perpendicularity as an undefined notion and lists its properties. Choquet defines orthogonal projection and the second axiom postulaes that if two segments of equal length but on different lines have the same endpoint then the orthogonal projections of each into the other have the same length. Choquet next chooses an inner product whose symmetry is guaranteed by his last axiom and having established a number of preliminary results, he shows that for all $x, y \in \Pi$, $d(x, y) = \|y - x\|$. At this stage Choquet's course is truly in the domain of Euclidean geometry.

In the remainder of the course Choquet deals with several other topics in geometry. He examines transformations of the plane and pays particular attention to the group of isometries I_o which fix a given point o and to the role of the abelian subgroup R_o (consisting only of rotations) of this group. This lays the groundwork for his definition of angle as rotation and so he obtains immediately that the set of angles with given vertex o is an abelian group. In order to measure angles, Choquet relies on the existence of continuous homorphisms from $\mathbb{R}$ onto the multiplicative group of complex numbers with absolute value one, having shown that the set of angles with given vertex is isomorphic to this group. He treats orientation algebraically and shows how an orientation of the plane can be obtained using either the group of affine transformations or the group of isometries. Choquet also treats elementary trigonometry and the geometric properties of the circle.

Papy

In Papy's opinion, linear algebra provides the best approach to geometry. In his course he uses synthetic axioms to help him represent the plane as a vector space. He begins with three axioms of incidence, then he defines parallelism and direction and his fourth axiom states 'Every direction is a partition of the plane'. At this stage, Papy gives his perpendicularity axiom. He now defines parallel projection as well as the notion of equipollence, which is extremely important in this course.

He proves that equipollence is reflexive and symmetric and by introducing the axiom 'Equipollence is transitive', he deduces that equipollence is an equivalence relation. The equivalence classes are called translations or vectors and the set of translations forms a group under composition. By fixing a point o in the plane Π, every point $x \in \Pi$ will define a vector $\vec{ox}$ and Papy proves that $(\Pi_0, +)$ is also a commutative group.

At this stage, an order axiom is introduced and now half-lines, half-planes etc. can be defined. Papy defines midpoints by using equipollence and so begins the important process of graduation of the line, which will integrate distance into his course. Using transitivity of equipollence and also midpoints Papy can lay off multiples of an interval of the form $p/2^q (p, q \in \mathbb{N})$ along a line. By inserting an archimedean axiom he makes sure he reaches beyond each point on the line and his continuity axiom ensures that every point of the line will be contained in one of his subgraduations. Papy also uses this process of graduation to build up the real numbers and he believes that by introducing the reals in this manner he not only enriches his Geometry but also the concept of a real number.

He defines the abcissa of a point on a line and uses this notion to define aaddition on the reals. Now it is possible to prove that $(\mathbb{R}, +)$ is a commutative group. Papy now defines homothetic maps and uses these to define the multiplication. He proves that $(\mathbb{R}, +, \cdot)$ is a field and that $\mathbb{R}_o$ is a real vector space. Next, Papy defines an inner product and the norm of a vector and so the distance between two vectors can be defined as $\|x - y\|$. We are now dealing with Euclidean Geometry and results such as Pythagoras's Theorem are easily proved using the vector space structure.

From here, Papy goes on to consider the classification of isometries. He discusses the group of angles and the isomorphism between this group and the group of rotations. (He defines angles as 'rotations which have lost their centres'). He also considers the field of similitudes, complex numbers and trigonometry.

Dieudonne

Dieudonne's geometry course is based completely on the concepts of linear algebra— he makes absolutely no concessions to synthetic methods. In fact his main reason for writing this book is to influence secondary school mathematics courses away from synthetic geometry towards a greater acceptance of linear algebra as a method of developing Euclidean plane geometry.

As Dieudonne's will be dealing with vector spaces over the real numbers, he begins by listing a set of axioms for $\mathbb{R}$ which is necessary and sufficient for his course. In particular, these axions make $\mathbb{R}$ into a totally ordered field. Even at this early stage his puritanism intrudes, because instead of a continuity axiom for $\mathbb{R}$, he uses an Intermediate Value property for quadratic

and cubic polynomial functions. This lack of a continuity axiom precludes the 'measurement' of angles in the usual sense, wich Dieudonne claims is rightly part of analysis and has nothing to do with algebra or geometry.

He now goes on to define a Eucidean Plane as a two-dimensional vector space over $\mathbb{R}$ with an inner product attached to it. All the standard affine results and properties (including axiom) can be deduced as theorems for the vector space axioms alone, with suitable definitions of line and parallelism. For example, denoting the vector space by E, for $0 \neq b \in E$ he defines a line as

$$L = \{a + \lambda b : \lambda \in \mathbb{R}\} = a + D, \qquad a \in E$$

where $D = \{\lambda b : \lambda \in \mathbb{R}\}$. is called the direction of L. Then $L_1 = a_1 + D_1$ and $L_2 = a_2 + D_2$ are parallel if and only if $D_1 \subset D_2$ (or vice versa). It can be shown that:

(i) For distinct L_1,L_2 lines in E: $L_1 \cap L_2 = \emptyset$ or $L_1 \cap L_2 = \{x\}$.

(ii) Given L_1 a line in E,$c \in E$, then there exists a unique line L_2 in E such that $c \in L_2$ and L_1 is parallel to L_2.

(iii) Through any pair of distinct points $c_1, c_2 \in E$ there is one and only one line.

Using the total ordering on $\mathbb{R}$, he can now define in an obvious way the concepts of midpoint of a segment, betweeness, half-line and line segment.

The standard definitions of translation and affine map are also introduced here viz. if E,F are two dimensional vector spaces, $a \in E$, then $t_a : E \to E$, $t_a(x) = a + x$ is a translation of E by a, while $u : E \to F$ is an affine map if $u = t_b \circ V$, where t_b is a translation of F ($b \in F$ arbitrary) and V a linear map from E to F. We get parallel projections by noting that any two distinct lines intersecting at the origin yield a direct sum decomposition of E (i.e. are supplementary subspaces) and so any $x \in E$ can be decomposed into the sum of two unique elements, one taken from each of the lines.

Placing an inner product on E now makes E into a Euclidean plane. We stress here that any inner product will do and that if two inner products are proportional ($\theta$is proportional to θ' if $\theta = \lambda\theta'$ for some $\lambda > 0$) then they both induce essentially the same Euclidean structure on E. This is not true for non-proportional inner products. Via the inner product we now have immediate access to the Eucidean concepts of orthogonality, perpendicularity, distance in the plane and angle, along with all the standard results from synthetic

Eucliden geometry. For example, with the usual definition of orthogonality, we can deduce a version of Pythagoras' theorem as a one line corollary of Minkowski's inequality, which itself is easy to prove in two dimensions. Two lines are perpendicular if their respective directions are orthogonal subspaces and, using a metric d induced by the inner product, we define a circle, for some fixed $x_0 \in E$ and $\lambda \in \mathbb{R}$, as a set $\{x \in E : d(x_o, x) = \lambda\}$. Dieudonne's treatment of plane geometry finishes with a glance at trigonometry and a development of complex numbers.

Because Dieudonne is intent on introducing linear algebra as well as geometry, some of his constructions are more elaborae than necessary if the main emphasis was on geometry. For example, his introduction of symmetry about the origin develops the concepts of eigenvalue, eigenvector and eigenspace, whereas in a geometric context we could simply define this symmetry map as $u : E \to E, u(x) = -x$. A treatment of plane geometry is given in [1], and this even manages to avoid the explicit introduction of vector space axioms by apprpriate definitions of addition and multiplication in $\mathbb{R}^2$.

This brief outline demonstrates clearly that Dieudonne's approach to geometry differs radically from he synthetic approach and consequently from the methods of treating plane geometry in most elementary school courses.

Conclusions

All three writers are agreed that the ideal way to approaxh geometry is via linear algebra. Consequently, they wish to arrive at a vector space structure as soon as possible. However, here Dieudonne disagrees with the approach adopted by both Choquet and Papy. Dieudonne claims there is no need to 'scaffold' from a synthetic to a vector space structure, and so operates immediately in a vector space.

Choquet and Papy adopt a similar type approach in their courses. They both begin with a set of basic (affine) acioms and graduate develop an algebraic structure on the plane by the addition of more synthetic axioms as required. However, there are areas of difference. For example, whereas Choquet assumes distance on a line and he real numbes, Papy uses graduation of the line to develop these concepts.

They are all agreed, however, that linear algebra gives us what Choquet calls a 'royal road' to geometry.

Before discussing the feasibiity of introducing geometry via linear algebra into second level school courses, it might be fruitful to outline some advantages

of such an approach. One fundamental advantage is that, with linear algebra, 'everything in elementary geometry can be obtained in a very straightforward manner by a few lines of trivial calculation' [3, p.10]. This is a powerful benefit, particularly when coupled with the fact that, in linear algebra, we have a theory 'where everything is ordered naturally around a few simple central ideas which also form the basis for later studies' [3, p.10]; after all, 'there are few mathematical concepts simpler to define than those of vector space and linear mapping' [3, p.11].

Another advantage is that linear algebra 'has become one of the most efficient and central theories of modern mathematics. Its applications now range over a wide and rich field, from the theory of numbers to theoretical physics, analysis, geometry and topology. Consequently there is great advantage to be gained from acquainting the young student at an early stage with the essential principles underlying this branch of mathematics' [3, p.10]. Closely allied to this point is that a linear algebra approach to geometry would bring second level mathematics courses more into line with university teaching [3, p.10].

To show that the advantages of applying linear algebra to geometry are not all one way, we should note that geometric concepts and constructions give 'life' to some of the 'drier' areas of linear algebra and so should make linear algebra more accessible to schoolchildren.

The final two advantages are inextriciably linked: 'From a mathematical point of view, the most elegant, mature and incisive method of defining a plane is as a two-dimensional vector space over the real numbers having an inner product' [2, p.14]. Along with this we have the that the concepts of vector space and inner product, with their developments, give us a logically perfect 'royal road' to geometry which we cannot afford to improve.

We will now look at the question of a linear algebra approach to geometry in schools. If we assume that geometry should be taught in secondary schools (either as part of the core curriculum or as an option extra) it is worth considering if we should remain wih the old synthetic (congruence) approach or whether a change to linear algebra would be beneficial. (Time constraints on the curriculum probably precludes a proper treatment of both.) In the course of this project we gathered some information on the second level geometry syllabuses of about eight countries (West Germany, Sweden, Belgium, France, England, Switzerland, Portugal and Canada) and in most cases (parts of France, in particular, being exceptions) it appeared that synthetic methods are still prefered, with scant and superficial regard given to linear algebra.

The main arguments against a linear algebra approach to plane geometry are outlined in Prof. Barry's article [0]: ' ... to subjugate geometry to linear algebra leads to an impoverishement of geometry. They (i.e. these who favour an old-fashioned (congruence) approach to geometry) value the visual as a helpful rewarding method of reasoning, they are reluctant for pedagogical reasons to impose extra unnecessary layers of abstraction on the young, and they value how mathematics can arise naturally in the small in geometry, growing from simple to more complex situations, in contrast with having to deal from the start with a large, abstract, complex system'. There are essentially two criticisms of linear lagebra here, which can be summarised as follows:

(i) An implicit criticism that the 'visual' is lost when linear algebra is applied to geometry.

(ii) That increased (and unnecessary) abstraction is unhelpful to the young.

We will examine these in order:

(i) The first thing we note is that the visual is not totally lost when we move over to linear algebra. Dieudonne himself recommends the use of instruments such as pantographs and affinographs to instil the idea of the 'geometric transformation of the plane or space as one entity'. He also suggests that the operations of vector addition and scalar multiplication in a two-dimensional vector space can be illustrated 'by a few months working wih squared paper' and this 'should be ample to familiarise pupils with the use of these (vector space) axioms and to prepare acceptance of the fact that the algebraico-geometrical edifice is founded on properties whose practical truth is empirically demonstrable.

However we feel that the whole question of visual aids to reasoning involving children between ages 13 or 14 and 17 or 18 shuold be examined more closely. There is a certain ambiguty in stressing the use of methods which encourage visual aids to reasoning whilst simutaneously telling children that diagrams in no way constitute a proof. The residual effects of this ambiguity are sometimes still apparent even at university. (If we look upon visual aids to reasoning as a subset of intuition, then the case of probability theory is applicable, where, if something is intuitively correct, it is most probably wrong). Clearly at primary level it is essential to use structures which are concrete and easily visualisable, but perhaps at the 1sat, 2nd and 3rd years in secondary schools we should begin to discreetly introduce abstract axiomatic systems, with the emphasis initially on concrete examples. (ii) We have

already touched upon this criticism in earlier parts of this paper. Clearly abstract systems cannot be introduced in the early primary years, but children in secondary schools are introduced to mny abstract concepts and seem to be able to deal satisfactorily with them.

The question of the 'necessity' of introducing abstract linear algebra to deal wih geometry in secondary schools is the very point at issue, and to deal with this properly would lead us into a critical discussion of synthetic methods, which would lead us too far afield. Suffice to say that Papy, Choquet and Dieudonne are convinced of the necessity.

A proper discussion of geometry a second level enevitably involves questions of mathematics and pedagogy. Whilst we have some little competence to deal with the former, we are com mpletely ill-equipped to deal with the latter. Our aim therefore is to raise questions and stimulate discussion, the ultimate outcome of which, we hope, will yield a course which will simultaneously satisfy the degree of rigour required by mathematicians as well as being accessible to children in secondary schools.

References

[1] P.D Barry, *Curriculum Development in Secondary School Mathematics, with Special Reference to Geometry*, IMS Bull. 18(1987), 78–91.

[2] P.D. Barry, *A Course in Plane Geometry*, Mathematics Department, University College, Cork.

[3] G. Choquet, *Geometry In a Modern Setting*,

[4] J. Dieudonne, *Linear Algebra and Geometry*,

[5] G. Papy, *Mathematique Moderne*, Didier, Brussels–Paris (1964–1967).

Department of Mathematics
University College Cork