Axiomatic Method and Independence Results[1]

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Ever since the discovery of non-Euclidean geometries, mathematicians were interested in formal methods and axiomatization of mathematical theories. It became apparent that ever present Euclidean geometry was not the only true geometrical reality, but that it could rather be substituted by other geometries, equally good and interesting on their own.

I will not exaggerate if I say that modern mathematics (by that I mean this century's mathematics) has been dominated by the use of formal i.e. axiomatic method. The aim of this article is to give a brief survey of axiomatic method with a few concrete applications.

Foundations of Geometry

There is little doubt as to whether the thirteen books of Euclid's "Elements" were the most valuable and influential scientific books of all time, if for nothing else but for the length of time during which they maintained their importance and influence in Mathematics, research and education. For over 2000 years the "Elements" were the standard of mathematical rigour, clarity and "absolute truth".

As it is always the case in scientific progress, however, there has never been room for contentment with any scientific achievement and this applied even to so "perfect" a work as Euclid's. Ever since the appearance of "Elements" there were questions about independence and consistency of Euclid's postulates; could any of the postulates (a more customary modern word for them is axioms) be derived as a theorem from the rest of the postulates? A special attention was paid to the famous fifth postulate:

[1]A lecture of a similar content was delivered by the author at the conference "Groups in Galway", Ireland, 09 - 10 May 1986.

Axiom E *For any plane π , any line $l \subset \pi$ and any point $A \in \pi \setminus l$ there exists at most one line k containing the point A and not intersecting the line l .*

This postulate was shown to be equivalent (given all other axioms) to the statement that there exists one rectangle or that the sum of the angles of a triangle equals to two right angles etc. Among the most penetrating mathematicians working on the subject were G. Saccheri (1667 - 1733) and J.H. Lambert (1728 - 1777) who have developed geometry arising from axioms without fifth postulate or its equivalents.

In the year 1829 the foundations of a great part of the 20th century mathematics (and we can safely say of the 20th century physics as well as arts) were established. Nikolai Ivanovich Lobachevski (1793 - 1856) published a paper [14] in which he developed a geometry that differed from Euclidean geometry by one axiom only. Namely it used the following negation of the fifth postulate E:

Axiom LB *For some plane π_0 , some line $l_0 \subset \pi_0$ and some point $A_0 \in \pi_0 \setminus l_0$ there exist at least two distinct lines k_1 , k_2 through the point A_0 that do not intersect the line l_0 .*

Great ideas appear in different great minds almost simultaneously: Janos Bolyai (1802 - 1860) had published in 1832 the same ideas in the appendix of his father's book (see [1]). Karl Friedrich Gauss (1777 - 1855) is said to have had investigations in the new geometry but he cannot be praised as much for the discovery not only because he did not publish any result of this kind but also because he had an entirely negative and discouraging attitude towards the discoveries of Janos Bolyai who abandoned mathematics at a young age after being exposed to such an attitude of the "King of mathematicians".

The theory was systematically built up according to strict deductive rules and had no inconsistencies. It was a big surprise at the time and there immediately arose questions as to whether the new geometry was as valuable as the ruling Euclidean geometry, in mathematics, philosophy and the physical world (the space measurements were taken with no instant success, only to be successfully performed after A. Einstein's work in relativity theory).

In the same period differential and projective geometries were developing. The first one led Bernhard Riemann (1826 - 1866) to the introduction (in 1854) of what is nowdays called Riemann spaces. Among them there stood out in particular spaces with constant curvature embracing the parabolic type that corresponds to the Euclidean space, the hyperbolic type corresponding to Lobachevski - Bolyai space and elliptic type corresponding to projective space with suitably chosen metric. Among the first interpretations of Lobachevski - Bolyai Geometry was the one given by Eugenio Beltrami (1835 - 1900). He used a pseudosphere to draw lines l and m, n (asymptotically converging to l) with $m \cap n = P$ and $m \cap l = n \cap l = \emptyset$, and thus interpreting them as "straight lines" obtained the LB axiom. Note that in this case the sum of the angles of a triangle is less than $180°$.

A similar interpretation of Riemannian Geometry is to be found in a model of a sphere, where great circles are interpreted as "straight lines" and thus they always intersect (in two points), and the sum of the angles of a triangle is greater than $180°$.

On the basis of Beltrami's ideas, Felix Klein (1849–1925) has given in 1871 in [13] basic results on consistency of Lobachevski-Bolyai geometry, whereas David Hilbert formally resolved problems of consistency of both Euclidean and Lobachevski - Bolyai geometry in [9] and [11] respectively.

Hilbert started with a set of *primitive notions* (non-defined intuitive notions such that new notions are built up of these). The primitive notions are: a set $\mathcal{S}$ ("space"), classes of subsets of $\mathcal{S}$ ("lines" and "planes"), ternary relation B and quaternary relation D on $\mathcal{S}$ (B : "betweeness", D : "equidistance"). (At the same time M. Pieri published in [16] and [17] two axiomatic systems of Euclidean geometry that each depended on only one primitive notion.)

Several statements (axioms) give properties of primitive notions that are most likely to be intuitively clear from "everyday experience". The axioms are usually grouped into: *Axioms of incidence* (stating set theoretical relations between points, lines and planes), *axioms of order* (listing properties of the relation B), *axioms of congruence* (about the relation D) and the *axiom of continuity* (enabling the Archimedean property). Geometry determined by these axioms is called *absolute geometry*. If the axiom E or LB is added to the axioms of absolute geometry, then we get respectively *Euclidean* or *Lobachevski-Bolyai geometry*.

It is also assumed that in constructing an axiomatic theory $\mathcal{T}$ use is made of other axiomatic theories (in our case set theory) which are presupposed (i.e. all their primitive notions and axioms are adjoined to those of $\mathcal{T}$).

The main demand on any axiomatic system is its *consistency* i.e. that no antinomy can be derived from the given set of axioms. The question whether some axioms can be shown to be the consequences of the others is that of *independence* of an axiomatic system. Though it may not be as important as

consistency, historically it is the investigation of independence of axioms that led to the discovery of new geometries. Proving consistency and independence consists of finding an (outside) consistent model satisfying the axioms (after certain interpretation) of the theory. An axiomatic system is *categorical* if any two of its models are isomorphic i.e. if it has a sufficiently strong axiomatic system determining uniquely its model up to an isomorphism.

The Euclidean geometry has been proved to be categorical and consistent provided the axiomatic system for arithmetic of real numbers is consistent. Namely the three-dimensional Cartesian space $\mathbf{R}^3$ has its own analytic geometry and the notions like points, lines, planes, betweenes and equidistance can be represented as ordered triples of real numbers and certain equations, together with a set and number-theoretical relations between them. Also the Euclidean fifth axiom E can be proved by proving that certain system of equations has a unique solution. The consistency of Lobachevski-Bolyai geometry was proved by interpreting it on Beltrami-Klein space – three-dimensional projective space with its analytic geometry (three-dimensional projective space is a quotient space of $\mathbf{R}^3 \setminus 0$ under the equivalence relation of proportionality of coordinates). The BL axiom holds in this model and it is obvious that Cartesian and Beltrami-Klein models are not isomorphic (they contain contradictory theorems E and LB respectively). Since both of these models are models for absolute geometry we conclude that absolute geometry is not categorical (since it contains at least two non-isomorphic models). On the other hand, as in the case of Euclidean geometry, Lobachevski-Bolyai geometry is also categorical.

We would like to emphasize here that the *consistency of Euclidean and Lobachevski-Bolyai geometry is only relative - dependent on consistency of the arithmetic of real numbers.*

For a detailed treatment of developing foundations of both geometries we recommend [2] to the interested reader.

Axioms for Set Theory

Methods used in a formal mathematical theory $\mathcal{T}$ are characterized by a very precise language, and, since I will content myself with the theory necessary for most of todays mathematics, namely set theory, I will call that language LST - the language of set theory. With LST we use the rules of logic (the axioms of first order logic, to be precise) and the rules for the formation of complex statements out of elementary ones. (This game must look strange to an outsider, since a non-mathematician friend of mine has recently told me that my mathematics is all squiggles together with lots of equality signs and zeros.)

The rules of the game are called axioms and, in the case of set theory, the most widely used system is the system of Zermelo-Fraenkel axioms (abbreviated as ZF); we give their heuristic list:

1. *Extensionality*: sets having the same elements are equal.
2. *Union*: the union of sets is a set.
3. *Infinity*: there is an infinite set.
4. *Power set*: the collection of all subsets of a given set is likewise a set.
5. *Foundation*: any non-empty set has a member disjoint from that set.
6. *Replacement Scheme*: for any set and a function with that set as domain, its image is also a set.

The replacement scheme as such is infinite and thus the list of axioms is infinite. Moreover it has been proved that no finite collection of LST sentences suffices to axiomatize ZF theory.

Though there may be various systems of axioms suitable for the same or different purposes, apparently not all of them are equally good or good at all. Every axiomatic system however should be tested by the following criteria:

(a) *Consistency*: $\mathcal{T}$ is consistent if there is no statement S such that both S and *nonS* can be derived from $\mathcal{T}$ or equivalently, if there is at least one statement S (formulated in the language of $\mathcal{T}$) that cannot be deducted from $\mathcal{T}$.

(b) *Completeness*: $\mathcal{T}$ is complete if, for every statement S formulated in the language of the theory $\mathcal{T}$, either S or *nonS* can be derived from the axioms of $\mathcal{T}$ according to the deduction rules.

(c) *Independence*: $\mathcal{T}$ has an independent set of axioms if none of its axioms can be derived from the remaining set of axioms.

As it is easy to conclude from the definitions just given that there is a relationship between consistency, completeness and independence, we note the following:

Proposition *A statement S is not provable in $\mathcal{T}$ if and only if $\mathcal{T}$ + nonS is consistent.*

Model Theory

As noted before, the first proofs on consistency and independence of a theory were given at the time of the appearance of non-Euclidean geometries by the use of models. The majority of the results on these metatheorethical questions in set theory were also achieved by the use of models; it is enough to get a (consistent) model for proving that a system of axioms is consistent. A very strong theorems in this area were given by Kurt Gödel (see e.g. [8]).

Gödel Completeness Theorem *If $\mathcal{T}$ is any consistent set of statements then there exists a model for $\mathcal{T}$ whose cardinality does not exceed the cardinality of the number of statements in $\mathcal{T}$ if $\mathcal{T}$ is infinite and is countable if $\mathcal{T}$ is finite.*

There is a Löwenheim-Skolem theorem very similar in content to the just stated theorem . One of the amazing consequences here is that there exists a countable family of sets with the property that if the membership relation is restricted only to those sets, then we get the model for the whole set theory (keep in mind that set theory contains uncountable sets and at the first sight it looks paradoxical that uncountable sets can be pictured in a countable model).

Gödel Incompleteness Theorem *If $\mathcal{T}$ is a consistent, sufficiently strong (i.e. if Peano arithmetic could be built in it), effective list of sentences (i.e. if there is an algorithm for recognizing a sentence from the list), then there is a statement S such that neither S nor nonS can be derived from $\mathcal{T}$.*

Gödel Underivability Theorem *If $\mathcal{T}$ is consistent, sufficiently strong, effective list of sentences, then $\mathcal{T} \nvdash \mathcal{C}on(\mathcal{T})$ (i.e. consistency of an axiomatic system cannot be proved from the axioms of that system alone).*

There are numerous applications of the methods used in model theory to the areas outside set theory. At the moment we give an example from non-commutative group theory.

Given a finite number of words $w_i = a_1^{n_1} \cdots a_{k_i}^{n_{k_i}} = 1$, $n_i \in Z$, $a_i \in G$ in a group G, can we determine (find an algorithm) whether another word $w = 1$? Or, equivalently, can w be obtained from w_i by taking multiplication, inverses and conjugation? This problem was translated into arithmetic terms and proved unsolvable in classical axiomatic system of set theory.

We give two more extremely important and useful results from model theory:

Theorem *If a theory $\mathcal{T}$ has an infinite model or arbitrary large finite models, then $\mathcal{T}$ admits models of arbitrarily large cardinalities.*

Compactness Theorem *If every finite subset of an axiomatic system $\mathcal{T}$ has a model, then the whole $\mathcal{T}$ has a model.*

The Axiom Of Choice And The General Continuum Hypothesis

From what has been said so far we understand that one cannot hope to base all conceivable mathematics on a single axiomatic basis and that is the reason that a continuous search for additional axioms is carried out. Various new axioms are being discovered every day. The space allowed makes it possible to list only the two most common ones: The axiom of choice and the general continuum hypothesis.

Axiom of Choice (AC): *For a given collection of sets, there is a set that contains one and only one element of each set of the given collection. This is equivalent to well ordering of any set as well as to the existence of infinite products.*

General Continuum Hypothesis (GCH): *For every infinite set X and every family $\mathcal{F}$ of subsets of X, $\mathcal{F}$ is in one-to-one correspondence either with a subset of X or with a set of all subsets of X. Using the aleph notation it is the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. If $\alpha = 0$, we have the Continuum hypothesis CH.*

Cantor used the axiom of choice as early as 1878 and the continuum hypothesis is also his [3]. Hilbert's first problem (see [10]) was the question of proving AC and CH from the system ZF. Both axiom of choice and the generalized continuum hypothesis however were proved to be independent of ZF. We list a few results from [4,8,5] ($Con(\mathcal{T})$ denotes consistency of $\mathcal{T}$, and $ZFC = ZF + AC$):

(1) $Con(ZF) \Rightarrow Con(ZFC)$,

(2) $Con(ZF) \Rightarrow Con(ZF + nonAC)$,

(3) $Con(ZF) \Rightarrow Con(ZFC + GCH)$,

(4) $Con(ZF) \Rightarrow Con(ZFC + nonCH)$,

(5) $ZF + GCH \Rightarrow AC$,

(6) $ZFC \Rightarrow$ there is a set of real numbers that is not Lebesgue measurable,

(7) $Con(ZF) \Rightarrow Con(ZF + nonAC+$ there is a set of real numbers that is not Lebesgue measurable$)$.

Whereas most mathematicians use AC in their work without questioning it, CH and GCH are not nearly as widely accepted. Moreover there are some very "natural" results following from the negation of GCH or CH.

Algorithmic Unsolvability

At the end of this survey I would like to point out a different kind of independence problems, yet closely related to the ones discussed in the previous section.

Ancient mathematicians have already noted that the ratio of the hypotenuse of an isosceles right triangle to its leg cannot be rational. They have also posed such questions as squaring the circle, doubling the cube or trisecting the angle by the use of only straight edge and compass. All these problems were shown to be impossible to solve, that is to say the axioms of the ruler and the compass do not suffice for making the required constructions (the problems were positively solved by the use of some more powerful devices...). It was also shown that there are polynomials already of the fifth degree whose roots could not be found by means of radicals (this last problem may had been the main step in the discovery of groups).

The most sophisticated among "modern" achievements of this kind is an ingenious solution (by Yuri Matijasevič in [15]) of Hilberth's tenth problem (see [10], also [6,7]).

Hilbert's tenth problem asked for an algorithm testing Diophantine polynomial equations for having integer (or, equivalently, natural) solutions. Although the notion of an algorithm can be precisely defined we will assume an intuitive feeling for the notion and, say that an algorithm is a "procedure" ("circulum vitiosus"!) that could be carried out by a computer in a finitely many steps and a bounded amount of time.

Definitions (a) A set S of ordered n-tuples $(a_1, \dots, a_n)$ of natural numbers is called *Diophantine* if for each such an n-tuple there is a polynomial $P(a_1, \dots, a_n, x_1, \dots, x_m)$, $m > 0$, with integer coefficients such that $P(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has a solution in natural numbers $x_1, \dots, x_m$.

(b) A set S of ordered n-tuples of natural numbers is *listable* (or, in a more latinized version, *recursively enumerable*) if there is a well defined algorithm for making a list of all members of S.

(c) A set $S \subseteq N$ is *computable* if there is an algorithm (of finitely many steps) for deciding whether any natural number belongs to S.

A few examples of Diophantine sets are as follows: integers having an odd divisor, the sets $\{(x,y) : x < y\}$, $\{(x,y) : x \text{ divides } y\}$ .... Some more examples can be obtained through the notion of a Diophantine function. It is such a function that its graph is a Diophantine set; or more precisely: a function f of n variables is a *Diophantine function* if $\{(x_1, \dots, x_m, y) : y = f(x_1, \dots, x_m)\}$ is a Diophantine set. The functions $T(n) = 1 + \cdots + n = n(n+1)/2$, $E(n,k) = n^k$, $F(n) = n!$, $B(n,k) = \binom{n}{k}$ are Diophantine.

It is easy to see that every Diophantine set is recursively enumerable. However the following fundamental result (see [15]) shows that the converse is also true:

Theorem *A set is Diophantine if and only if it is recursively enumerable.*

If we express this theorem in, for us more suitable "polynomial form", we have:

Main Theorem *There is a procedure that can be used on any algorithm listing a set S of n-tuples of natural numbers, to get a polynomial P with integer coefficients such that $P(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has a solution in nonnegative integers $x_1, \dots, x_m$ if and only if $(a_1, \dots, a_n) \in S$.*

Now, if a set S is computable it is recursively enumerable but a basic result in recursion theory states that the converse is not true:

Theorem *There is a listable set $S \subseteq N$ which is not computable.*

Corollary *There is a polynomial $P(a, x_1, \dots, x_m)$ such that there is no algorithm for deciding whether $P(a, x_1, \dots, x_m) = 0$ has integral solutions in $x_1, \dots, x_m$, for any given values of the integer parameter a.*

This is a strong negative solution to Hilbert's tenth problem since it states that there is no algorithm for testing solvability of Diophantine equations, even with one parameter only.

Notice that the result does not give the way to find out which Diophantine equations are indeed algorithmically solvable.

Let me now mention a few positive results. One of the consequences of the Main Theorem above is that there exists a polynomial P with integer coefficients containing all prime numbers among its values (there are various examples of such polynomials of less than 12 variables and polynomials of the kind of the fifth degree). For the novelty's sake we list one of such polynomials (see [12]) containing "only" 325 symbols:

Theorem *The set of primes is exactly the positive range (as the variables range over natural numbers) of the following polynomial of the 25th degree and 26 variables :*

$$
\begin{aligned}
&P(\textit{the letters of the English alphabet}) = \\
&\quad (k+2)\{1 - (wz + h + j - q)^2 - [(gk + 2g + k + 1)(h + j) + h - z]^2 \\
&\quad -[16(k+1)^3(k+2)(n+1)^2 + 1 - f^2]^2 - (2n + p + q + z - e)^2 \\
&\quad -[e^3(e+2)(a+1)^2 + 1 - o^2]^2 - [(a^2-1)y^2 + 1 - x^2]^2 \\
&\quad -[16r^2y^4(a^2-1) + 1 - u^2]^2 - [(a^2-1)l^2 + 1 - m^2]^2 - (ai + k + 1 - l - i \\
&\quad -[((a + u^2(u^2 - a))^2 - 1)(n + 4dy)^2 + 1 - (x + cu)^2]^2 \\
&\quad -(n + l + v - y)^2 - [p + l(a - n - 1) + b(2an + 2a - n^2 - 2n - 2) - m]^2 \\
&\quad -[q + y(a - p - 1) + s(2ap + 2a - p^2 - 2p - 2) - x]^2 - \\
&\quad -[z + pl(a - p) + t(2ap - p^2 - 1) - pm]^2\}
\end{aligned}
$$

It is worth mentioning that the methods discovered can be used to reformulate some of the classical problems in mathematics, by getting equivalent statements saying that certain polynomial Diophantine equations have no solutions in nonnegative integers. Among such classical problems are the last

Fermat's theorem, Goldbach's conjecture, four color problem, Riemann conjecture etc. (However the twin-primes conjecture cannot be reduced in this way).

Finally we state here a strong version of Gödel's incompleteness theorem. One can show that the set of theorems in a formalized mathematical theory is listable, whereas the set of unsolvable Diophantine equations is not. Precisely we have:

Incompleteness Theorem *Let T be a theory with its language containing symbols $0, S, +, *, <$, with the following properties:*

(a) *T is consistent ,*

(b) *T is listable ,*

(c) *T is strong enough to prove any of the statements of the forms : $\alpha+\beta=\gamma$, $\alpha*\beta=\gamma$, $\alpha<\beta$, where α, β, γ are among 0, $S0$, $SS0, \ldots$, with S being the successor function. Then there is a (polynomial) Diophantine equation $P(x_1,\ldots,x_m)=0$, determined by T, such that $P=0$ does not have a solution in natural numbers, but we cannot prove it within the theory T.*

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MATHEMATICAL EDUCATION

Approaches To School Geometry

Mary Morrin David Murphy Ann O'Shea

Introduction

This article arises from a postgraduate course in geometry given by Professor Barry at U.C.C. As part of the course we undertook some project work on the geometry courses of Georges Papy, Gustave Choquet and Jean Dieudonne. Here we hope to review these three courses and their potential for inclusion in the secondary school curriculum.

First of all, we must ask the question: why teach geometry? One obvious reason for teaching geometry is its application to real life situations and problems. Through the study of geometry children develop practical skills in such areas as measurement, calculations of areas and volumes, use of grids and co-ordinate systems. It also gives them an understanding of the concepts of two-dimensional and three-dimensional space. Clearly geometry has application to topics in mathematics and can indeed be regarded as a unifying theme in the mathematics curriculum. It provides a rich source of visualisation for arithmetical and algebraic concepts. Geometry is essential for mastering calculus and therefore all other fields that have calculus as a prerequisite. A major reason for the inclusion of geometry in the secondary school curriculum is its value as a vehicle for stimulating and exercising general thinking skills, skill in deductive reasoning and problem solving. Through its precise use of language, geometry can also play a part in the development of skills in communication. Therefore, geometry has an important role in the secondary school curriculum.

The next question is: How should we teach geometry in secondary schools? It seems to us that there are two main approaches. One the one hand there