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On The Level

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We present a survey on the notion of the level of a field and its various generalizations. We describe a lot of results that are attractive from an algebraic viewpoint and also highlight the extremely interesting relations between algebra and topology that have been unearthed in the last decade in connection with the level. We hope to persuade the reader that this is an appealing area of mathematics and that it should be a fruitful area for future research. In Section 1, we look at levels of fields, in Section 2, we deal with commutative rings and the link with topology and in Section 3, we look at the non-commutative situation and generalisations of the idea of level.

1 Fields

Let F be a field. F is said to be *formally real* if -1 is not expressible as a sum of squares in F. If F is not formally real we define the *level* of F, denoted $s(F)$, to be the smallest natural number N such that -1 is a sum of N squares in F (We define $s(F) = \infty$ if F is formally real).

The Artin-Schreier theorem [35, p.227] says that a field F is formally real if and only if F admits an ordering (i.e. $s(F) = \infty$ if and only if F admits an ordering).

We look now at levels of some well-known fields

Example 1 $F = \mathbf{R}$, the real numbers, $s(\mathbf{R}) = \infty$.

Example 2 $F = \mathbf{C}$, the complex numbers, $s(\mathbf{C}) = 1$ since $-1 = i^2$ in $\mathbf{C}$.

Example 3 $F = \mathbf{F}_p$, a finite field with p elements, p an odd prime. It is a fairly easy exercise to show $s(\mathbf{F}_p) = 1$ if $p \equiv 1 \pmod 4$ and $s(\mathbf{F}_p) = 2$ if $p \equiv 3 \pmod 4$.

Example 4 $F = \mathbf{Q}_p$, the field of p-adic numbes. Then $s(f) = 1$ if $p \equiv 1 \pmod 4$, $s(F) = 2$ if $p \equiv 3 \pmod 4$. If F is the field of dyadic numbers then $s(F) = 4$. See [35, p.151] for a proof.

Example 5 For F an algebraic number field $s(F) = 1, 2, 4$, or ∞. This is known as Siegel's theorem as it was first proven in [58]. See [35, p.300] for a modern proof.

The notion of level seems to have first explicitly arisen about sixy years ago following the work of Artin and Schreier on formally real fields [4] and Artin's solution of Hilbert's seventeenth problem [3]. It is implicit however in earlier work of Hilbert, Landau and others. Indeed the general problem of representing integers as sums of squares seems to have perpetually engaged the attention of mathematicians, e.g. the work of Diophantos, Fermat, Lagrange, Gauss and numerous others. The German word 'Stufe' was used for the level and this is the reason for the notation $s(F)$.

The suggestion that $s(F)$ is always a power of two if finite was made in [64] and in 1934 H. Kneser [34] proved that $s(F) = 1, 2, 4, 8$ or a multiple of 16. For a period of almost thirty years, no significant advance was made on this although a few authors did examine the level [62, 31, 63, 38]. The major breakthrough came in 1936 when Pfister, inspired by a colloquium lecture at Gottingen by Cassels on sums of squares in rational function fields, succeeded in proving that $s(F)$ must be a power of two if it is finite. (See [51] for Pfister's own account of his discovery). Pfister also succeeded in producing examples of fields of prescribed level 2^k for each positive ineger k. Before this no example of a field of level greater than 4 had been known. Pfister's work appears in [48] and uses quadratic form theory.

It is easy to see that -1 is a sum of squares in the field F if and only if the $(n+1)$-dimensional quadratic form given by the identity matrix is isotropic, i.e. represents zero non-trivially. Thus there is an obvious connection between quadratic form theory and the notion of level of a field. Pfister introduced the notion of a quadratic form being multiplicative and this was the key idea needed to obtain his results on the level of a field. Accounts of his results may be found in [35, Ch.11] and in [56, Ch.2, §10 and Ch.4 §4]. A very quick proof that $s(F)$ is a power of two if finite appears in [56, p.69–72], this proof being a simplification due to Witt of the original proof.

Various authors have obtained results about the levels of specific kinds of fields. Pfister in [50] showed that $s(F) \leq 2^d$ for F non-real and of transcendence degree d over a real closed field. In the realm of number theory the level of cyclotomic fields was studied in [16], [17], [45]. For algebraic number fields in general $s(F) = 1$, 2, or 4 if finite by the theorem of Siegel mentioned earlier. The question of how to distinguish between the cases $s(F) = 1$, $s(F) = 2$, $s(F) = 4$ has been examined in [27, 19, 8]. We quote for example the following theorem of [27].

Theorem *Let F be an algebraic number field. Then $s(F) \leq 2$ if and only if F is totally imaginary and the local degrees at all primes extending the rational prime 2 are even.*

See also [44], [53] for some further results. The question of how the level $s(F)$ is related to other field invariants has been considered. Let $q(F)$ be the cardinality of $\dot{F}/\dot{F}^2$, the group of square classes. Pfister [49] showed that $q(F) \geq 2^{k(k+1)}/2$ where $s = 2^k$ and this was improved by Djokovic [25], using an argument involving graph theory, who showed that for $s > 2$, $q(F) \geq 2^{s+1}/s$ where $s = s(F)$. See also [35, Ch.11] for more information.

2 Commutative Rings

The definition of level is meaningful not just for fields but for any ring with identity element 1. The ring need not be commutative. We deal with commutative case in this section and the non-commutative case in Secion 3. (One could even discuss the level of a non-associative ring with identity but this has not been considered by anyone to the author's knowledge).

Results on $s(R)$ for R the ring of algebraic integers in a p-adic field were obtained by Riehm [55] who showed $s(R) = 1, 2$, or 4 in this case. For R being the ring of algebraic integers in the algebraic number field K results on $s(R)$ were obtained in [26, 46, 47 and 44]. In particular $s(R) \leq 4$ when $s(K) < \infty$ is proved in [46], and in [26] it is proved that $s(R) = 1$ if $s(K) = 1$, $s(R) \leq 3$ if $s(K) \leq 2$ and $s(R) \leq 4$ if $s(K) \leq 4$, this theorem being attributed to M. Kneser. For further information see the above references.

It is easy to see that the level of commutative ring need not always be a power of two.

Example $R = \{0, 1, 2, 3\}$ with addition and multiplication modulo four. Then $s(R) = 3$ because $-1 = 3$ and 1 is the only non-zero square in R.

Knebusch [32] proved that $s(R)$ is a power of two when R is a local ring in which 2 is a unit. Baeza followed this up by proving the same result for semi-local rings with 2 a unit and had more results on levels of rings in [5], [7, app.1], and [6]. In particular he proved in [6] that, for a Dedekind domain R with field of fractions F, $s(F) \leq s(R) \leq 1 + s(F)$.

A major landmark in the theory of levels occurred in 1979 when Dai, Lam and Peng, [24] proved the following:-

Theorem *Any positive integer may occur as the level of a commutative ring.*

The sensational feature of their work was that they proved this theorem by appealing to a theorem from topology. Their proof goes as follows:-

Let

$$R = \frac{\mathbb{R}[x_1, \ldots, x_n]}{(1 + x_1^2 + x_2^2 + \cdots + x_n^2)}$$

i.e. the quotient of the Polynomial ring $\mathbb{R}[x_1, \cdots, x_n]$ by the ideal generated by $1+x_1^2+x_2^2+\cdots+x_n^2$. Clearly $s(R) \leq n$ and the problem is to show $s(R) < n$ is impossible. Suppose -1 is a sum of $n-1$ squares in R. Then there exist polynomials $p_j(x_1, x_2, \cdots, x_n), j = 1, 2, \cdots, n-1$ and $q(x_1, x_2, \cdots, x_n)$ such that

$$-1 = \sum_{j=1}^{n-1} p_j^2 + q\left(1 + \sum j = 1^{n-1} x_j^2\right)$$

The trick is to replace $x = (x_1, x_2, \ldots, x_n)$ by $ix = (ix_1, \ldots, ix_n)$ where $i^2 = -1$. Then we may write $p_j(ix) = r_j(x) + is_j(x)$, r_j and s_j being real polynomials, r_j being even, i.e. $r_j(-x) = r_j(x)$, and s_j being odd, i.e. $s_j(-x) = -s_j(x)$.

Now define a map $f : S^{n-1} \longrightarrow \mathbb{R}^{n-1}$ by

$$f(x) = (s_1(x), s_2(x), \ldots, s_{n-1}(x)) \text{ for each } x \in S^{n-1}$$

Since f is continuous we may apply the Borsuk-Ulam theorem from topology [59,p.266] which says that there must exist a pair of antipodal points of S^{n-1} mapped to the same element of $\mathbb{R}^{n-1}$ i.e. $f(z) = f(-z)$ for some $z \in S^{n-1}$. But $f(-x) = -f(x)$ for all x because each s_j is odd and thus $f(z) = 0$ i.e. $s_j(z) = 0$ for each j. This implies that $-1 = \sum_{j=1}^{n-1} r_j(z)^2$, i.e. -1 is a sum of squares in $\mathbb{R}$, completing the proof by contradiction.

After the Dai-Lam-Peng paper had appeared algebraic Borsuk-Ulam theorems were proven by Arason-Pfister [2] and also by Knebusch [33]. The theorem of [2] goes as follows:-

Theorem *Let $f_1, f_2, \ldots, f_{n-1}$ be a set of polynomials in $x = (x_1, x_2, \ldots, x_n)$ with coefficients in a real closed field F. Assume the f_j are odd, i.e. $f_j(-x) = -f_j(x)$. Then there exists $z \in S^{n-1}$ for which $f_j(z) = 0$ for all j.*

It is easy to show that the above theorem is equivalent to the statement that given a set of polynomials $f_1, \ldots, f_{n-1}$ in x there exists $z \in S^{n-1}$ with $f_j(-z) = f_j(z)$ for each j. Write each f_j as an even plus an odd polynomial!

This algebraic Borsuk-Ulam theorem for polynomials in fact will yield the full Borsuk-Ulam for continuous functions by using the Weierstrass approximation theorem and the compactness of S^{n-1}.

Proof We briefly outline the proof. Introduce an extra indeterminante x_0 and multiply each monomial in f_j by a suitable power of x_0 so as to make a homogeneous polynomial $\tilde{f}_j$. Replace x_0^2 by $x_1^2 + x_2^2 + \cdots + x_{n-1}^2$ (x_0 appears in even powers because f_j is odd) and obtain f_j which are homogeneous polynomials of odd degree in $x_1, x_2, \ldots, x_{n-1}$. Now aplying a theorem of Lang [36] these polynomials must have a common zero in F^n which we may take to be in S^{n-1}. (Dividing by $\sqrt{\sum x_i^2}$ is all right as they are homogeneous!)

Dai and Lam [23] investigated in much greater detail the links with topology that had been forged in [24]. They discovered that the level in algebra is closely related to notions in topology that had been considered earlier by C.T. Yang [65, 66] and by Conner and Floyd [20, 21]. We describe this now.

Let $(X, -)$ be a topological space equipped with an involution $-$, i.e. a continuous map $X \longrightarrow X$, $x \rightarrow \overline{x}$ of a period two (so that $\overline{\overline{x}} = x$)

Example 1 $X = S^n$, $-$: the antipodal map.

Example 2 $X = \mathbb{C}$, $-$: complex conjugation.

Example 3 $X =$ the Stiefel manifold $V_{n,m}$ of orthonormal m-frames in $\mathbb{R}^n$, with involution ε_r givenby

$$\varepsilon_r(v_1, \ldots v_r, v_{r+1}, \ldots v_m) = (v_1, \ldots, v_r, -v_{r+1}, \ldots, -v_m)$$

An equivariant map between $(X, -)$ and $(Y, -)$ is a continuous map $f : X \longrightarrow Y$ such that $f(\overline{x}) = \overline{f(x)}$ for all $x \in X$. The *level* of the space $(X, -)$ is then denoted $s(X, -)$ and is defined by

$$s(X, -) = \inf\{n : \text{there exists an equivariant map from } (X, -) \text{ to } (S^{n-1}, -)\}$$

Essentially the same invariant as $s(X, -)$ had been studied earlier in [20, 21] where it was called the co-index.

The link with algebra is obtained by associating to $(X, -)$ the ring of all equivariant maps from $(V, -)$ to $(\mathbb{C}, -)$. This ring is denoted A_X.

Theorem (Dai, Lam [23]) $s(X,-) = s(A_X)$

They also define the *colevel*

$s'(X,-) = \sup\{n :$ there exists an equivariant map from$(S^{n-1},-)$ to $(X,-)\}$

Motivated by this topological notion of co-level one may define for any real algebra A an algebraic *co-level*

$s'(A) = sup\{n :$ there exists a real algebra homomorphism from A to $A_{S^{n-1}}\}$

It is easy to see that, for any $(X,-)$, $s'(X,-) \leq s'(A_X)$. Dai and Lam proved [23] that if X is a real affine variety then $s'(X,-) = s'(A_X)$.

Another interesting and related notion examined in [23] is that of the *sublevel* of a commutative ring R, denoted $\sigma(R)$. We say $\sigma(R) = n$ if $0 = a_1^2 + a_2^2 + \cdots + a_{n+1}^2$ for elements $a_1, a_2, \ldots, a_{n+1}$ such that the ideal generated by these $(n+1)$ elements is the whole of R and N is the least integer for which this property holds. One notes that $\sigma(F) = s(F)$ for any field F and that $\sigma(R) \leq s(R)$ for any R. The simplest example of a ring R where $\sigma(R) \neq s(R)$ seems to be

$$R = \frac{\mathbb{Q}[x,y]}{(1 + x^2 + 2y^2)}$$

for which it can be shown that $\sigma(R) = 2$ but $s(R) = 3$. See [23] and [15] for proof.

If $s(R) = 1, 2, 4$, or 8 it can be shown that $\sigma(R) = s(R)$ by using the 2-square, 4-square or 8-square identities [29, p.417].

For a commutative ring in which 2 is a unit it is not too hard to show [23] that $s(R) = \sigma(R)$ or $1 + \sigma(R)$. The following natural question was posed and answered in [23].

Which pairs (n,n) and $(n,n+1)$ occur as $(\sigma(R), s(R))$ for some R? They showed that (n,n) occurs for all n and that $(n,n+1)$ occurs for all $n = 1, 2, 4$ or 8. They exhibited examples for all these cases. For $n(n,n+1)$ their examples are the rings

$$\frac{\mathbb{R}[x_1, x_2, \cdots, x_{n+1}, y_1, y_2, \cdots, y_{n+1}]}{(1 - \sum x_i^2, 1 + \sum y_i^2, \sum x_i y_i)}$$

To prove $\sigma(R) = n$ and $s(R) = n+1$ involves relating $\sigma(R)$ and $s(R)$ to the level of certain Stiefel manifolds and calculation of the level of these appeals to non-trivial topological results. (In particular Adams' result on the non-existence of elements of Hopf invariant one). We refer the reader to [23] for the details. There are many more interesting connections with topology in [23], in particular using results on equivariant maps into Stiefel manifolds. For example Adams' theorem on vector fields on spheres may be used to show that for any commutative ring R, if the form $n \times <1>$ over R represents -1 then in fact the form $n \times <1>$ contains $\rho(n) \times <-1>$ as an orthogonal summand. (Here $\rho(n)$ is the Hurwitz-Radon number [35, p.131]).

We should also mention one question raised in [23] and still unsolved at present, namely the Level Conjecture. Let C be a commutative ring and

$$R = \frac{C[x_1, x_2, + \cdots + x_n]}{(1 + x_1^2 + \ldots + x_n^2)}$$

The Level Conjecture is that $s(R) = n$. For $C = \mathbb{R}$ we have described the proof and Arason-Pfister [2] have proved it when C is any field. It is not clear what technique to use for an arbitrary commutative ring C.

Recently much progress has been made on the study of levels in connection with real algebraic geometry. The following lemma is a starting point for some of this theory.

Lemma *Let R be a commutative ring with 1. Then $s(R) < \infty$ if and only if $s(F(R/\wp)) < \infty$ for all prime ideals $\wp$ of R, $F(R/\wp)$ denoting the field of fractions of the integral domain $R/\wp$.*

Proof See [18], [12] or [22] where it was first observed. See [18] for how this leads to the Real Nullstellensatz and Positivstellensatz in real algebraic geometry.

When R is the co-ordinate ring of an affine variety V without any real points Mahé has succeeded in finding a bound for $s(R)$ in terms of the Krull dimension of R (One may show easily that V has no real points if and only if $s(R) < \infty$).

Theorem *Let F be a real closed field and A an $F-$algebra of finite type with Krull dimension d, specA having no real points. Then $s(A) \leq d - 1 + 2^{d+1}$.*

Proof See [43]. This theorem answers question 11.3 posed in [23].

We finish this section by pointing out that levels are only one aspect of the general study of sums of squares. Throughout the history of mathematics sums of squares have been a topic of fascination and curiosity. Some general references are [28, 61]. One particular problem is that of the *Pythagoras number* $p(R)$ for a commutative ring R. We define $p(R)$ to be the least integer n such that every sum of squares in R is a sum of at most n squares. The determination of $p(R)$ is generally a very difficult problem. See [14, 15], for further information and references. One may also examine k-th powers instead of squares and can generalize the level by asking for the least n such that -1 is a sum of n k-th powers. (k should be even as it is trivial for odd k). See [10, 9] for information on this for fields, also [30] for rings.

3 Non-Commutative Rings

There is very little in the literature about levels or sums of squares in the non-commutative situation. The following theorems were proved recently.

Theorem (Leep, Shapiro and Wadsworth) *Let D be a division algebra finite dimensional over its centre F. Then the following three statements are equivalent:*

(i) 0 is a non-trivial sum of squares in D;

(ii) -1 is a sum of squares in D;

(iii) each element of D is a sum of squares in D.

Proof See [37]. Note that if D is a field this theorem is an easy exercise.

A quadratic form q over a field F is *weakly isotropic* if, for some n, the orthogonal sum of n copies of q is isotropic.

Theorem *Let D be a division algebra finite dimensional over its centre F. Then 0 is a non-trivial sum of squares in d if and only if the trace form of D is weakly isotropic.*

(Note: the trace form is the map $q : D \longrightarrow F$, $q(x) = \text{tr } x$, tr being the reduced trace [56, p.296].

Proof See [39].

It follows that $s(D) < \infty$ if and only if the trace form of D is weakly isotropic for D as in the above theorems. In [40] we examined the case of D being a quaternion division algebra and obtained the following results.

Theorem *There are quaternion division algebras D wih $s(D) = 2^k$ for any k and with $s(D) = 2^k + 1$ for any k.*

(It is an open question whether or not other integer values can occur as $s(D)$ for quaternion algebras.) The examples with level 2^k and $2^k + 1$ are described as follows: Let $F = K((t))$, the Laurent series field in one variable t and let $K = \mathbb{R}(x_1, x_2, \ldots, x_n)$, the rational function field in $x_1, x_2, \ldots, x_n$. Let $D = \left(\frac{a,t}{F}\right)$ where $a = \sum_{i=1}^n x_i^2$, i.e. D is the quaternion algebra defined by $i^2 = a, j^2 = t$ etc. For $n = 2^k+1$ it is shown that $s(D) = n$. For $n = 2^k$ we use $F = K(t)$, the rational function field, K as above, but let $D = \left(\frac{-t, t-a}{F}\right)$ and then it turns out that $d(D) = n$. Our techniques make use of Pfister's results on products of sums of squares.

One may also consider sublevels for non-commutative rings and a few results appear in [41], mainly for quaternion algebras.

From one point of view it may be argued that the appropriate generalization of sums of squares to the non-commutative case is sums of products of squares. For example Szele [60] proved the following generalization of the Artin-Schreier theorem.

Theorem *Let D be any skewfield. Then D admits an ordering if and only if -1 is not a sum of products of squares in D.*

This suggests one possible generalization of level to what we will call the *product level* and denote $s_\pi(R)$ for any ring R. The *product level* $s_\pi(R)$ is the least integer n such that -1 is a sum of n products of squares in r. Define $s_\pi(R) = \infty$ if -1 is not a sum of products of squares in R. Szele's theorem thus may be rephrased as $s_\pi(D) = \infty$ if and only if D admits an ordering.

Not also that Albert [1] proved that an ordered skew-field must be infinite-dimensional over its centre and thus $s_\pi(d) < \infty$ for finite dimensional algebras.

The only result in the literature on s_π is the following due to Scharlau and Tschimmel.

Theorem *Every positive integer can occur as $s_\pi(d)$ for some skewfield D.*

Proof See [57].

The examples prduced in [57] are all infinite-dimensional over their centre so it would still seem open as to whether or not $s_\pi(D)$ can take every positive integer value for skew-fields finite over their centre.

We now make a few elementary observations about $s_\pi(D)$ for skewfields D.

(1) It is possible to have $s(D) = \infty$ but $s_\pi(D) = 1$. An example of this appears in [37]. Let $D = \left(\frac{x,y}{F}\right)$ where $F = \mathbb{Q}(x,y)$ the field of rational functions in two variable x, y over the field $\mathbb{Q}$ of rational numbers. Then $s(D) = \infty$ since the trace form of D is the form $< 1, x, y, -xy >$ which is not weakly isotropic over F. However $s_\pi(D) = 1$ since this is true for any quaternion algebra as $-1 = i^2j^2((ij)^{-1})^2$.

(2) Any commutator $[x,y] = xyx^{-1}y^{-1}$ in D is a product of squares since $[x,y] = x^2(x^{-1}y)^2(y^{-1})^2$. Thus whenever D contains a pair of elements which anti-commute then $s_\pi(D) = 1$. In particular $s_\pi(D) = 1$ for any quaternion algebra D.

(3) Let D be finite-dimensional over its centre F and let N be the reduced norm map D to F. See [56, p.296] for a definition of N. If D is odd-dimensional over F then $s_\pi(D) > 1$ provided -1 is not a square in F.

Proof $N(-1) = -1$ for n odd whereas if $-1 = x_1^2x_2^2 \ldots x_n^2$ then $N(-1) = N(x_1)^2N(x_2)^2 \ldots N(x_n)^2$ is a square in F. Contradiction.

Theorem *If D is even-dimensional over F then $s_\pi(D) = 1$ if the reduced Whitehead group $SK_1(D) = 1$*

Proof $SK_1(D) = \{x \in D : N(x) = 1\}/[D,D]$ so if $SK_1(D) = 1$ every element of norm 1 is a product of commutators and hence a product of squares. $N(-1) = 1$ for even-dimensional D so that $s_\pi(D) = 1$.

It seems quite likely that $s_\pi(D) = 1$ always for D even-dimensional over its centre but we cannot prove this.[1]

(4) Let $\dot{D}$ be the multiplicative group of non-zero elements of D and $\dot{P}$ the normal subgroup of all non-zero products of squares. Then $s_\pi(D) \leq |\dot{D}/\dot{P}|$, the index of $\dot{P}$ in $\dot{D}$.

[1] A. Wadsworth has just proved this!

Proof Let $-1 = \sum_{i=1}^{n} p_i$, each $p_i \in \dot{P}$, n minimal. Let $b_k = \sum_{i=1}^{k} p_i$ for each $k = 1, 2, \ldots, n$. Then $b_1, b_2, \ldots, b_n$ give distinct cosets of $\dot{P}$ in $\dot{D}$ since n is minimal. Thus $s_\pi(D) \leq |\dot{D}/\dot{P}|$.

We describe now another possible generalization of level which can be defined in both the commutative and non-commutative cases.

Let R be a ring with an identity and equipped with an evolution, i.e. an anti-automorphism of period two. We use the symbol to denote an involution. A *hermitian square* in R is an element of the form $\overline{x}x$ for some $x \in R$.

We define the *hermitian level* of the ring with involution $(R, -)$ to be the least integer n for which -1 is a sum of n hermitian squares in $(R, -)$. We write $s_h(R, -)$ for the hermitian level and define $s_h(R, -) = \infty$ if -1 is not expressible as a sum of hermitian squares. Note that $s_h(R, -)$ depends on the involution $-$ and not just the ring R. The general idea for obtaining results on $s_h(R, -)$ is to use hermitian form theory in the same way that quadratic form theory is used to obtain results on the usual notion of level. This is done in [42] and we summarise some of the main results obtained there.

(1) Any positive integer n may occur as $s_h(R, -)$ for a ring with non-trivial involution. Specifically $s_h(RC_3, -) = n$ where R is a ring with $s(R) = n$, such as described in Section 2, C_3 is the cyclic group of order three and $-$ is the involution on the group ring RC_3 induced by $\overline{g} = g^{-1}$.

(2) Let $(R, -)$ be either a field with involution or a quaternion division algebra with standard involution. Then $s_h(R, -)$ is a power of two if it is finite.

(3) Let $D = \left(\frac{a,b}{F}\right)$, the quaternion division algebra with non-standard involution $\hat{}$ defined by $\hat{\imath} = -i, \hat{\jmath} = j$. If F is real closed or p-adic then $s_h(D, \hat{}) = 1$. If F is a number field then $s_h(D, \hat{}) = 1, 2,$ or ∞.

(4) It is an open question as to whether or not $s_h(D, \hat{})$ must always be a power of two for quaternion division algebra in general. We show in [42] that every power of two occurs as $s_h(D, \hat{})$ and also that integers of the form $2^k - 1$ cannot occur but we have no other results on this question.

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A Sociological Question

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Write on John H. White's theory of "open" and "closed" Catholicism, in the context of religion in modern Irish society.

— from a Maynooth BA exam paper.

Let OP be the set of all possible opinions. When endowed with Archdeacon Wellbeloved's aggiornamento topology (the topology of substantial agreement on the broad fundamentals of the question), OP becomes a completely regular connected Hausdorff topological space. Regrettably, OP satisfies neither the first nor the second axiom of countability, and hence is non-metrizable, but then you can't have everything. The space OP contains non-contractible loopy sets of opinions, and hence is not simply-connected. The problems this poses may sometimes be overcome by passing to the universal covering space, the space of all idee-fixed homotopy classes of circular arguments, also known as full socio-loopy space.

A *person* is a set-valued function p, defined on the set $\mathbb{R}$ of all real numbers, with values in the power set of OP. The majority of persons ordinarily encountered have the additional property that $p(t)$ is empty before an initial conception-time, depending on the person (depending on some other persons, too, who enjoy it a lot more). It is also usually found that $p(t)$ remains constant once t exceeds about 15 years after conception-time. The technical term for this is that $p(t)$ has been *set in concrete*.

Let N denote the set of propositions contained in the Nicene Creed.
Let A denote the set of propositions contained in the Apostle's Creed.
Let I denote the singleton: { The Pope is tops }.

Definition A person p is a *catholic* at time t if and only if $p(t)$ contains the union of N, A, and I.

Evidently, a catholic is open at time t if it holds a neighbourhood of each of its opinions. It is closed if it holds an opinion x whenever it holds opinions arbitrarily close to x.