IRISH MATHEMATICAL SOCIETY BULLETIN

The aim of the BULLETIN is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears three times each year, in March, September and December.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. Authors who have access to TEX are invited to submit their articles in the form of TEX input files (plain TEX and LATEX are equally acceptable). Typed articles will be given the same consideration as articles submitted in this form.

Correspondence concerning the Bulletin should be addressed to:

Irish Mathematical Society Bulletin
Department of Mathematics
University College
Galway, Ireland

Correspondence concerning the Problem Page should be sent directly to the Problem Page Editor at the following address:

Faculty of Mathematics
Open University
Milton Keynes, MK7 6AA, UK

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THE IRISH MATHEMATICAL SOCIETY

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Vice-President	Dr. Fergus Gaines	Department of Mathematics University College Dublin
Secretary	Dr. Richard Timoney	School of Mathematics Trinity College Dublin
Treasurer	Dr. Gerard M. Enright	Department of Mathematics Mary Immaculate College Limerick

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EDITORIAL

It is with great pleasure that we bring you this first Galway issue of the Bulletin. Our first task must be to thank Donal Hurley, Pat Fitzpatrick and Martin Stynes for the magnificent work they have done during the years the Bulletin has been produced in Cork. We hope that the high standards set by them will be maintained by us. Some continuity will be ensured by the fact that Phil Rippon has agreed to continue with his editorship of the excellent Problem Page.

As the reader will no doubt already have noticed, we have taken the opportunity of the change to adopt TEX as the typesetter for the Bulletin. Apart from the ease which which it can handle almost any conceivable mathematical text, and the pleasing appearance of the pages, we foresee a situation in which a substantial proportion of the articles in the Bulletin will have been typeset in TEX by the authors, and the resulting document sent to Galway in the form of a TEX input file, either through HEANET or on a disk. Indeed, two of the articles in this issue have been transmitted to us in this way. Apart from the fact that this greatly expedites the production process, the author enjoys far greater control over the final appearance of his or her work. However, we wish to give an assurance that any articles submitted in the traditional typed form will be given exactly the same consideration as those in TEX.

We wish to thank the members of the Mathematics department of UCG, who have helped in many ways. A particular word of gratitude is due to the departmental secretary, Carol Conroy, who took on so successfully the rather daunting task of learning LATEX. Thanks are also due to Richard Timoney for making his expertise in TEX available to us whenever we required it. All of the faults, however, are entirely the work of the Editor and Associate Editor.

IRISH MATHEMATICAL SOCIETY

Ordinary Meeting

April 16, 1987

An ordinary meeting was held at 12.15 pm at the Dublin Institute for Advanced Studies. The President, S. Dineen, was the Chairman. There were 8 members present.

1. The minutes of the meeting of December 19th, 1986 were approved and signed.

2. The President reported on various decisions tken by the Committee, as follows:

(i) It was decided to solicit opinions from all interested parties with a view to formulating a plan for the directions in which Mathematics in Ireland should develop. Such plans have formed the basis for significant improvements in other countries — the David report in the USA and a more recent Griffiths report have been an effective basis for improving Federal funding of Mathematical research in the US, for example.

Anybody with opinions on how Mathematics (in Ireland) should develop or adapt to changing demands should communicate their views in writing to the Secretary by the end of 1987. It was agreed that the Secretary and the President would also solicit views from appropriate industrial and commercial sources.

(ii) The new editors of the Bulletin have decided to use the TEX typesetting system and would welcome suitable articles, in machine-readable form, prepared using TEX . Authors intending to avail of this facility should ideally consult the editors in advance of typing their paper.

(iii) The Committee agreed to hold a meeting of the Society at UCD in September, 1987, if arrangements could be made in time by the Vice-President, F.J. Gaines. Part of the funding was available from sources in UCD.

3. The Secretary explained a plan agreed by the Committee to ask all Mathematics Departments at HEANET sites to set up an electronic address for the Department. Such an address could be used to send electronic mail to any individual in the department (including those with unknown or non-existent personal electronic addresses). Departments are being asked to set up this with a standard usename MATHDEP, if at all possible.

4. It was agreed that it would not be appropriate for the Society to lend its support to the new International Campaign which has replaced the former Massera, Scharansky and Orlov campaigns (these have achieved their goals). The new campaign has the objective of bringing an end to "all torture, abduction and oppression by agents of the Pinochet regime in Chile".

Richard M. Timoney,
Secretary

MEMBERSHIP LIST SUPPLEMENT

The following supplement to the 1986 Membership List was compiled from the Treasurer's records on the 25th of September 1987.

Additions

87247	Flood R.	Dept. for External Studies, University of Oxford, 1 Wellington Square, Oxford OX1 2JA, England
87248	Gow R.	Dept. of Mathematics, UCD, Dublin
87249	Reilly M.	Dept. of Statistics, UCD, Dublin
87250	Paramhans S.	B, 1/148–1–Ka, Assi, Varanasi 221005, India
87251	Morrison J.	Dept. of Mathematics, Towson State University, Towson, Maryland 21204, USA
87252	Byrne A.	College of Technology, Kevin Street, Dublin
87253	O'Brien B.	College of Technology, Kevin Street, Dublin
87254	Milligan G.	College of Technology, Kevin Street, Dublin
87255	O'Brien E.	Dept. of Mathematics, Research School of Physical Sciences, Australian National University, GPO Box 4, ACT 2601, Australia
87256	Carroll F.	Dept. of Mathematics, Ohio State University, 231 West 18th Ave., Columbus, Ohio 43210–1174, USA
87257	Dowling J.	Dept. of Math. Physics, Campus Box 390, University of Colorado, Boulder, Colorado 80309, USA
87258	Lambert A.	Dept. of Mathematics, University of North Carolina, Charlotte, NC 28223, USA
87259	Kim Jin Bai	Dept. of Mathematics, College of Arts and Sciences, West Virginia University, Morgantown, West Virginia 26506, USA

87260	Conneelly M.	Dept. of Mathematical Physics,UCG, Galway
87261	Cullen H.	Dept. of Mathematics, University of Massachusetts at Amherst, Lederle Graduate Research Center, Amherst, MA 01003, USA
87262	Hurley S.	Dept. of Mathematics, NIHE, Limerick
—	McLoughlin J.	Dept. of Mathematics, Maynooth College, Co. Kildare (Student Member)

Amendments

85020	Kelly E.G.	Dept. of Biostatistics, Columbia University, New York, Ny 10027, USA (ex UCC)
85038	Barry M.	Dept. of Mathematics, Allegheny College, Meadville, PA 16335, USA (ex Carysfort College)
85134	O Sé D.	Regional Technical College, Carlow (ex Maynooth)
85161	Grone R.	Dept. of Mathematical Science, San Diego State University, San Diego, California, CA 92182, USA (ex Auburn U.)
85163	Harary F.	Box 3 CU, New Mexico State University, Las Cruces, NM 88003, USA
85171	Porter T.	U.E.R. de Mathematiques, Universite de Picardie, 33 Rue St. Len, 80039 Amiens Cedex, France
86168	Burns J.	College of Technology, Kevin Street, Dublin (ex Maynooth)
86205	Lynch P.	Meteorological Service, Dept. of Communications, Glasnevin Hill, Dublin 9

NEWS

Personal Items

- **Professor Leroy Beasley** of Utah State University in Logan, Utah, is spending a sabbatical year in the Mathematics Department of UCD. Professor Beasley works in Linear Algebra and Finite Group Theory.
- **Professor Jerome Sheahan** of the Statistics Department, University of Alberta, Edmonton, is spending a sabbatical year in the Mathematics Department of UCG. Professor Sheahan works in Statistical Analysis and Probability Theory.
- **Roger Dodd** is currently on leave of absence from the School of Mathematics of TCD; he is visiting Hiroshima University.
- **Graham Ellis** has been appointed to a permanent position in the Mathematics Department of UCG. Dr. Ellis works in Algebraic Topology, Homological Algebra and Algebraic K-Theory.
- **Pat Fitzpatrick** is on leave of absence from the Mathematics Department of UCC for 1987/88. He is spending the year at the University of Toulouse, working with the group there on Algebraic Coding Theory.
- **Ciaran Murphy**, formerly of UCG, has joined the Statistics Department of UCC. His particular interests are Operations Research and Information Systems.
- **Russell Higgs** has been appointed to a permanent position in the Mathematics Department, UCD.
- **Pól Mac Aonghusa** has been appointed to a temporary position in the Mathematics Department of Maynooth College.
- **Philip Murphy** has recently taken up a Department of Education Postdoctoral Fellowship at TCD.

- **Martin Newell** is on sabbatical leave from UCG for 1987/88. He is presently visiting the University of Padua.
- **Aongus Ó Cairbre** has joined the staff of the College of Commerce, Rathmines, Dublin.
- **Noel Gorman** has taken up a temporary position in the School of Mathematics of TCD. Dr. Gorman is on leave from the Dublin Institute for Advanced Studies.
- **Daniel O'Regan** has been appointed to a temporary position in the School of Mathematics, TCD.
- **Anthony K. Seda** is on leave of absence from UCC for 1987/88. He will be visiting Bristol University, Imperial College London and Edinburgh University to join the groups there working on formal programming and the use of formal methods in the specification and verification of software.
- **Eamonn O'Brien**, a former UCG student, now studying at the Australian National University, Canberra, was awarded the B.H. Neumann Prize for the best student lecture at the Annual Meeting of the Australian Mathematical Society in May 1987 for his talk "A Computer Based Description of 2–Groups".
- **Mícheál Ó Searcóid** was awarded a Ph. D. in Mathematics at UCC in June this year. His supervisor was Professor Robin Harte. Dr. Ó Searcóid has now taken up a permanent position in the Mathematics Department of UCD.
- **Martin Stynes** of the Mathematics Department, UCC, has been invited to give a keynote lecture at the BAIL V Conference in Shanghai in June 1988.

The Irish Mechanics Society

A meeting of the Irish Mechanics Group was held in UCD in September in conjunction with Campus Ireland. Participants came from Belgium, Canada and Ireland and eleven papers were presented. It was decided to rename the Group as The Irish Mechanics Society, a constitution for which was formally drawn up and ratified. The first official meeting of the new Society will be a conference in UCC in May 1988.

Irish Winner in International Contest

First place in the LOGO Division of the International Computer Problem Solving Contest this year was awarded to 11 year old John Farragher, a sixth class pupil at St. Paul's Primary School in Limerick and a participant in the Mary Immaculate College Computer project for children of high ability in Mathematics.

The International Contest is organised annually by the University of Wisconsin and it mainly involves computer programming problems for second level students. The inclusion of a LOGO Division for children under thirteen in this year's Seventh Annual Contest reflects the ever-increasing use of computers and the particular interest in LOGO in primary schools. A LOGO Division for the under sixteen age group is planned for inclusion in the 1988 Contest.

John was selected when Dr. Pat O Sullivan asked all school principals in Limerick City to send two or three sixth class children of higher than average mathematical ability to the College for testing. Dr. O Sullivan was organising the special LOGO project for mathematically bright children for Mary Immaculate College which is conducted in association with St. Patrick's College Drumcondra. John was one of the eighteen children who came out of the screening process and undertook an eleven-week intensive LOGO course taught by Pat O Sullivan and Dr. Gerard Enright. In April he and fellow pupil Ryan Meade entered as separate teams at the Dublin venue of the International Contest.

The contest was in the form of a two-hour practical examination held at the Holy Faith School in the Coombe and organised for the Computer Education Society of Ireland by Mr. Michael Brady. Using Commodore 64 computer equipment, John solved the five problems with which he was presented in

about an hour and a half and he spent the remaining thirty minutes checking his solutions. John Farragher won that event, Ryan Meade was placed third and John's entry was sent to the University of Wisconsin for ranking amongst 370 teams from all over the United States and from several other countries. A few weeks later everyone associated with John and with the Irish section was delighted to hear that he had won first place in the world.

The Computer Education Society of Ireland has been active for many years in the promotion of the use of computers in schools and in recent years it has paid particular attention to encouraging such development at primary level. Mary Immaculate College, through its Department of Mathematics and Computer Studies, is also playing a central role in this area of curriculum development. The College provides courses not only for its own undergraduate students but also for practising teachers in the Mid-West Region. It also undertakes a major programme of experimental research with local school children and it has initiated a one-year teachers' course for a Diploma in Computer Studies which is acting as a very effective catalyst in primary school activities.

The Harte's A Wonder

Marcel Dekker recently published "Invertibility and Singularity for Bounded Linear Operators" by Robin Harte, price $119.50 (ISBN 0-8247-7754-9). We quote from the author's own description of the contents:

> The Doctor tells all: Normed spaces as you have never seen them before.
> Out of the Closet: Almost open mappings and the boring truth behind the Open Mapping theorem.
> Enlargements: Linear operators laid bare.
> Compact Operators: Small but perfectly formed.
> Fredholm Operators: Algebra is no laughing matter.
> Almost Exactness: A contradiction in terms?
> Joint Spectra: Joseph Taylor and his technicolour dream-coat ...

The book will be reviewed in a future issue of the Bulletin.

Summary of Results of the 1987 Irish National Mathematics Contest

The ninth Irish National Mathematics Contest was held on Tuesday, 3 March, 1987 and attracted more contestants this year than we've had for some years. In all, 1,832 participants from 81 schools took part, compared with 1,342 from 75 schools last year.

As in all previous years, the paper for this year's INMC was set by the MAA Committee on American Mathematics Competitions for the 38th Annual American High School Mathematics Examination, which was taken by up to 400,000 to the MAA Committee for allowing us to use their examination materials.

The new scoring system that was introduced last year was in operation again this year. This system, which is intended to reward intelligent guessing and discourage random guessing, also tends to yield higher marks of 85 marks in this year's INMC than in previous years. Altogether, 234 students scored 85 or more marks this year, while, of these, 128 scored 90 or more marks, more than twice the number who fell into this category last year. A score of 100 or greater was achieved by 22 students. We are very pleased with the enthusiastic response from the schools to our invitation to enter students in this year's contest and with the high standard attained by such a large number of contestants.

The highest score in this year's INMC was obtained by Alan Conway, High School, Rathgar, Dublin 1. Alan got 118 marks, an excellent score. Score of 114 were achieved by Patrick Browne, also of the High School, and Fergal Byrne of St. Benildus College, Kilmacud Road, Dublin 14.

The highest team score—the sum of the three highest scores of individual contestants from the same school—was achieved by St. Benildus College with a score of 328. The High School in Rathgar was second with 321 and O'Connell School, Dublin 1, a close third with 319. The winning team was composed of Fergal Byrne, Garrett O'Neill and Damian Lawlor.

A prize-giving ceremony will be arranged early in December to honour the top scorers in the INMC.

Future issues of the Bulletin of the Irish Mathematical Society and the Newsletter of the Irish Mathematics Teachers Association will carry more information about the contest.

The Fifth Irish Invitational Mathematics Contest was held on Tuesday, March 24, 1987. To keep the numbers down to manageable proportions only those with a score of 90 or higher marks in the INMC were invited to participate in the IIMC, exam materials for which were also supplied by the MAA Committee on American Mathematics Competitions. The contest was especially difficult this year, the hardest it has been since its inception five years ago. The top scorers were: Fergal Byrne of St. Benildus College, Dublin, John O'Brien of Presentation College, Cork, and Andrew Farrell, a twelve-year old from Navan, who each scored 5 out of a maximum of 15.

Finbarr Holland, University College, Cork.
Tom Laffey, University College, Dublin.

IMS MEMBERSHIP

The Ordinary Membership subscription for the session 1987/88 is £5. Payment is now overdue and should be forwarded to the Treasurer without further notice.

Institutional Membership is available for 1987/88 at £35. Such support is of great value to the Society. Institutional Members receive two copies of each issue of the Bulletin, and may nominate up to five students for free membership.

Should people be Paid to Do Research in Mathematics?

Most mathematicians will admit that there is a sense of achievement in having made a mathematical discovery, no matter how small. This emotion along with the desire to gain the respect of other mathematicians are the inner motivating factors for doing mathematical research. A more materialistic motive for doing research is the fact having a number of papers published lessens the likelihood that one will be unemployed for long. Thus research mathematics shares this motivation with many other occupations: if you're good at it, you'll feel good about it and you'll get a good job out of it. It does not follow from this personal motivation that working as a research mathematician is necessarily a good thing.

Mathematics is an important tool in most of natural science and engineering. Clearly one would expect mathematicians to be paid for their services in the same way as laboratory technicians and bricklayers are. But there is a difference between applying the results we know and trying to find out new results. The questions begs itself: do we need to know any more mathematics — can't we make do with what we have?

There is no reason to suppose that mankind will perish without further mathematical research. This does not mean that some future mathematical discovery might not be useful; nor does it deny that a mathematician who has done some "pure" research will usually be a better applied and teaching mathematician.

However, given that technology has outstripped man's needs, if not her desires, research mathematics is becoming a luxury good. Its benefits to mankind in general are increasingly marginal and we face the question: are we justified in paying mathematicians to do research and attend conferences when over half the world's adults are illiterate?

Brendan McCann,
Department of Mathematics,
University College Galway.

Dr. Martin J. Newell (1910–1985)

Seán Tobin

The signature Máirtín O Tnúthail appears prominently on the very fine silver salver which was presented to President Éamon deValera, to mark his Golden Jubilee as Chancellor of the National University of Ireland, in December 1971. The salver is now on display in the Presidential Room of the National Museum—and it must have been very agreeable to Éamon de Valera, who maintained a lifelong interest in mathematics, to have on his memorial salver the signatures of two mathematicians (the other being that of Dr. Donal McCarthy, President of U.C.C.) as pro-vice-chancellors of the University. Dr. Newell's signature is firm and very clear, and this fits well with salient characteristics of the man himself: clarity of expression and firmness of decision, two qualities which became especially significant during his tenure of the Presidency of University College Galway, from 1960 to 1975, when the course of university development was charted for many years to come.

Martin J. Newell was born and bred in the heart of Galway, where his family lived in Shop Street, one of the old central streets which still preserve the outlines of the mediaeval City. He was educated there in St. Joseph's College and in 1926 he entered University College Galway, taking first places in the County Council and University Entrance Scholarships. A brilliant career as a student was crowned with the award in 1930 of the M.Sc. Degree in Mathematical Science (with first-class honours), and the N.U.I. Travelling Studentship. This brought him to Cambridge for three years in St. John's College, where he studied for the Mathematical Tripos.

In 1933 he was appointed to the staff of St. Michael's College in Listowel, and in 1935 he returned to Galway as Lecturer in Mathematics (through Irish), in succession to Eoghan McKenna who had become Professor of Mathematical Physics. Incidentally his own successor in Listowel was James Callagy B.A., they had been fellow-students at U.C.G., where they shared an enthusiasm for geometry.

In 1950 Martin J. Newell was appointed a member of the Governing Board of the School of Theoretical Physics at D.I.A.S., and he continued in that capacity until 1965. In 1952 he was awarded the degree D.Sc. by the N.U.I.

for his published work, and in that year also was elected a member of the Royal Irish Academy.

In 1955 he succeeded Michael Power as Professor of Mathematics in U.C.G.; he himself had no direct successor since — on his recommendation — the College extinguished the monolingual Lectureship in mathematics and established a regular Lectureship instead. About this time also a successful operation restored his hearing (he had been troubled for some years by an increasing deafness) and so in 1955 he entered on a new phase of life. This was a time when the Irish university system, dormant during World War II and its aftermath, was itself quickening to a new life. New courses were planned, young research workers were recruited, and in general the staff in universities began to exert more influence on policy. Symptomatic of this was the formation of staff associations, U.C.G. being well to the fore. In all of the new initiatives Martin Newell played some part, and he was chairman of Cumann Lucht Teagaisce an Choláiste when that body arranged the presentation of a bronze bust to Monsignor Pádraig de Brún, to mark his retirement as President of the College. (This fine portrait bust, the work of Cork sculptor Séamus Murphy, is now in the U.C.G. Staff Club.)

An even greater change was to occur in 1960 when Martin Newell, a surprise candidate, succeeded Pádraig de Brún as President. He was the first native of Galway to hold that office, and he brought to it a high sense of purpose and integrity — in private he said that his guiding principle was "Only the best is good enough for U.C.G.!" His lively sense of humour and ready wit combined with unfailing courtesy, which made him welcome company in any social gathering, helped also to lighten the burden of the many committees which he was called on to chair.

His Presidency coincided with a period of great expansion in the Irish economy, and so he was able to bring to fruition plans already begun under Monsignor de Brún for major new developments. His period of office, from 1960 to 1975, might well be termed the Golden Age of U.C.G. Student numbers more than tripled, extensive new lands were purchased for the campus, a long-range physical development plan was established, a new Library and academic complex was built, staff numbers increased greatly and many new disciplines were introduced.

Dr. Newell was well-regarded also in the larger academic community, and in 1971 was awarded the honorary degree of Ll.D. by Dublin University in recognition of his achievements. He maintained his interest in mathematics, and took a partial Sabbatical from his administrative duties in order to prepare

publications [7], [8] and [9].

After his early retirement in 1975, for health reasons, his tall spare figure with its shock of wavy hair was only occasionally seen in U.C.G., in summer-time - winters were spent in Spain, summers in his lakeside house on Upper Lough Corrib, which was appropriate for a lifelong angling enthusiast. It was with a deep sense of loss and sadness that friends and colleagues learned of his sudden death, soon after his return to Spain in the Autumn of 1985. His body was brought home to rest in his native city; the large and distinguished attendance at funeral ceremonies in Dublin and in Galway was eloquent testimony to the affection and respect which he had earned over his many years of service to his country.

No appreciation of Brod Newell, as he was known to his friends, could be complete without a tribute to his wife Noreen who is happily still hale and hearty, and whose friendship is treasured by those of us who have had the privilege of knowing her for many years. She too is blessed with a keen sense of humour, and was always the soul of hospitality to students and staff alike. Her constant support and her careful concern for his health contributed greatly to her husband's success as teacher, researcher and administrator; and she created a happy family life for their five children (Sinéad, Michael, Martin L., Éamonn and John).

Since the present healthy state of algebra studies in Ireland has arisen largely because of the impetus given by M.J. Newell and his students in Galway it is pleasant to record that his son, Martin L., has followed his example, being himself a professor in the U.C.G. department of mathematics and well-known internationally for his research work in group theory.

A Personal Note

As a student majoring in Mathematical Science in U.C.G., I helped Brod Newell to proofread some of his papers—actually, as it happens, in the same room where I am writing these lines. It was due to his influence that I went to Manchester to study algebra (in fact group theory with Graham Higman); and at his request I returned to Galway from the U.S.A. to become the first Lecturer in Mathematics at U.C.G. I owe more than I can say to him and to Noreen for continuing friendship, advice, encouragement and hospitality over the years since then. One of his old colleagues, whom I asked for his considered opinion of Brod, said "He was always a gentleman". I think that this is as good a summing-up as one could wish for. Ar dheis Dé go raibh a anam dílis.

Mathematical Work

Martin J. Newell won the Peel Prize in Geometry at U.C.G. in 1926, and in later years kept a hopeful eye on subsequent winners as potential future mathematicians. It seems fitting that his first venture into print was as one of the contributors to Mathematical Note No. 1031, on *A difficult converse*, in the 1932 Mathematical Gazette [1]. This note commences as follows: "The difficulty of proving that if the [internal] bisectors of two angles of a triangle are equal [in length] then the triangle is isosceles is well known; three fairly simple proofs are given in this note." The first proof is Newell's: quite a simple argument based on an ingenious construction—possibly a carryover from his student days in U.C.G. Incidentally, the other two proofs were derived from articles in the *Lady's and Gentleman's Diary*, 1859 and 1860, showing that the comment "well known" was well founded.

Curiously enough, despite his inclination towards classical geometry Martin Newell's own forte was in combinatorial algebra. He was a master of matrix theory, a virtuoso performer of polynomial calculations. These qualities are evident in—and indeed are the basis for—his published research papers [2] to [9]. Some of these interests show through also in his book *Algébar Iolscoile* ("A University Algebra") [10] where Laplace and Cauchy expansions jostle for space with theorems of Jordan and Binet-Cauchy on compound matrices, in a beautifully concise exposition.

Papers [2] to [7] inclusive are in the classical tradition of group representation theory deriving from Frobenius and Schur; they consist in the main of original and often elegant proofs of known results, most of which are due to Littlewood or Murnaghan. In some cases these results are extended or generalized, and methods are given for simplifying tedious calculations which arise in their development.

References [A] and [B], as well as the later [C], give the background to Newell's papers; as texts they make difficult and at times frustrating reading but, for those who are interested in sampling the original flavour of the Frobenius - Schur results, a beautiful introduction is given in Walter Ledermann's book [E]. Boerner [D] covers roughly the same ground as [A] and [B], but this material is no longer the sole concern of texts on group representation theory;

for instance Walter Feit's recent book [F] ignores it, being concerned entirely with modular theory.

In the years 1948-50 Martin J. Newell wrote up and published, in five papers [2] to [6], results which he had obtained over the previous decade. The first of these is in some ways the most elegant of the series; he commenced by remarking that while the quotient of two alternant determinants had been much used, the quotient of two alternant *matrices* had not been exploited – an omission which he proceeded to rectify, by showing "that consideration of the quotient matrix furnishes simple proofs for most known theorems".

Some definitions will be useful, to explain the thrust of his work. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a sequence of n indeterminates and let $(t) = (t_1, t_2, \ldots, t_n)$ be a strictly decreasing sequence of non-negative integers $t_1 > t_2 > \cdots > t_n \geq 0$. Let $A(t_1, \ldots, t_n)$ be the alternant matrix whose (i, j) entry is $\alpha_i^{t_j}$. The determinant $|A(t)|$ is an alternating polynomial in $\alpha_1, \ldots, \alpha_n$; in particular when $t_1 = n-1$ it is Vandermonde's determinant. Let σ_r and h_r respectively be the elementary and the complete homogeneous (Wronski) symmetric polynomials of weight r in the indeterminates $\alpha_1, \ldots, \alpha_n$ where $r \geq 0$; define $h_i = \sigma_i = 0$ if $i < 0$. The key result in [2] is the following:

For any positive integer s, and for $1 \leq k \leq n$,

$$\alpha_k^s = [h_{s-n+1}, h_{s-n+2}, \ldots h_s] Q [\alpha_k^{n-1}, \alpha_k^{n-2}, \ldots, \alpha_k, 1]'$$

where the dash denotes transposition and Q is the matrix

$$q_{ij} = (-1)^{j-i} \sigma_{j-i} \,. \qquad (\text{Thus } |Q| = 1).$$

From this it follows immediately that

$$A(t_1, t_2, \ldots, t_n) \equiv BQA(n-1, n-2, \ldots, 1, 0)$$

where B is the matrix $b_{ij} = h_{t_i - n + j}$.

Taking determinants we see that

$$\frac{|A(t_1, t_2, \ldots, t_n)|}{|A(n-1, \ldots, 1, 0)|} = |B| \,.$$

This is the classical Jacobi-Trudi equation, and that is Newell's proof of it.

Now clearly $|B|$ is a symmetric homogeneous polynomial, with integer coefficients, in $\alpha_1, \alpha_2, \ldots, \alpha_n$. If $\lambda_i = t_i + n - i, 1 \leq i \leq n$, then $|B|$ has degree

$\lambda_1 + \lambda_2 + \cdots + \lambda_n = m$ say, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The sequence $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ gives a non-increasing partition of m; reversing the procedure we might now, given such a partition (λ) define $t_i = \lambda_i - (n - i)$ thus getting a strictly decreasing sequence (t). The corresponding polynomial $|B|$ is known as the *Schur function*, or S-function, associated with (λ) and is denoted here by the symbol $\{\lambda\}$. These functions $\{\lambda\}$ play a central role in studies of the characters of the symmetric groups S_m, as also in the character theory of the real orthogonal and sympletic groups.

The study of identities which link S-functions and other basic symmetric functions such as σ_i, h_i and the power-sums $s_i = \alpha_1^i + \cdots + \alpha_n^i$, the study of rules for expressing given symmetric functions of $\alpha_1, \ldots, \alpha_n$ in terms of S-functions— hence e.g. rules for calculating the coefficients $g_{\lambda\mu\nu}$ where $\{\lambda\}\{\mu\} = \sum_\nu g_{\lambda\mu\nu}\{\nu\}$: these are the subject matter of the papers [2] to [7]. Possibly the strongest influence of this work is to be seen in Murnaghan's book [C] based on a course of lectures which he gave in 1957 at the Dublin Institute for Advanced Studies. In the preface he refers to considerable improvements upon the exposition in [A], and mentions "for instance, the treatment of the modification rules for the rotation, sympletic and orthogonal groups, in which I have been able to use with great profit the ideas of Professor M.J. Newell."

Already two years earlier, in his lectures to the 1955 St. Andrew's Colloquium, Philip Hall had cited Newell's work — I am indebted to Ian Macdonald of Q.M.C. for this reference. Hall was discussing the proof of certain key properties of Schur functions and remarked that "a particularly elegant derivation of the central theorem, and of many other important formulae" had been given a few years previously by Dr. M.J. Newell. (In fact Newell was present at that Colloquium, and must have been pleased with this complimentary reference to [2]. Subsequently however he expressed more interest in the fact that, having gone to speak to Hall after one of the lectures, he had noticed on the margin of Hall's manuscript a pencilled note "Joke here" followed apparently by an outline. He was surprised that the eminent Cambridge group theorist should (a) think a joke necessary during his lecture and (b) need to write one down. This illustrates a fascinating aspect of Newell's own character, namely the way in which he combined apparently contradictory traits. Thus he himself, quick in repartee and a good raconteur, would never have needed to write down a joke — yet on the other hand he never, to my knowledge, made a joke when lecturing.)

The last paper in this series, [7], deserves special mention, being Newell's only joint paper (his co-author was a long-time associate, Rev. Professor

James McConnell of D.I.A.S.); it was written and published during his term of office as president of U.C.G., it makes effective use of the concept of "conjugate symmetric functions", and it was written in order to prove fully certain results given by Littlewood [B] where the proofs, "when looked into ... are found to be incomplete". It is a substantial paper, filling a substantial gap, and represents perhaps a final act of pietas.

The papers [8] and [9] also appeared while M.J. Newell was President, and exhibit the working-out of ideas which had interested him previously. In [8] he shows how to express the discriminant (necessarily non-negative) of the characteristic equation of a real symmetric $n \times n$ matrix S as a sum of squares of minors from a rectangular $n \times n^2$ matrix M such that $m_{rs} = \text{trace}(S^{r+s-2})$; for $n = 3$ this reduces to an old theorem due to Kummer (1843). The final paper [9] is an application of algebra to a problem in calculus. If $F(x_1, x_2, \ldots, x_n)$ is a continuously differentiable function of n real variables, then the nature of a stationary point is (h opefully) determined by a quadratic form; the matrix A of this form is $n \times n$ with $a_{ij} = \partial^2 F / \partial x_i \partial x_j$, and whether the form is semi-definite or not may be decided directly by examining the signs (+ or −) of the leading principal minors of A. An analogous (and apparently new) criterion for the case where there are $n-k$ auxiliary conditions $\varphi_i(x_1, \ldots, x_n) = 0$, $1 \leq i \leq n-k$, is derived very neatly in [9], and here again determinantal expansions due to Jacobi and Laplace are used to great effect.

M.J. Newell took his responsibilities as Lecturer through Irish very seriously, and prepared textbooks on algebra, calculus and geometry. These were written for An Gúm, the Government agency for publication of books in Irish; the algebra and calculus texts were published but An Gúm apparently found the geometry too unorthodox and it never appeared in print. The calculus text [11] had Michael Power, then Professor of Mathematics at U.C.G., as co-author and was quite orthodox. The most noteworthy book was Algébar Iolscoile [10], which was used regularly by honours classes in U.C.G. It is a brief introduction to classical algebra with a minimum of jargon and a maximum of information, and is still treasured by those who acquired it as their first university-level text in algebra.

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ARTICLES

Simulated Annealing — New Developments in Combinatorial Optimization

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Introduction

Combinatorial Optimization (CO) may be defined as the problem of maximizing (or minimizing) some measure of utility (the objective function) of a system with a large but finite number of states. There may be constraints which limit the domain which must be searched for the optimal solution. Perhaps the best known example is Integer Linear Programming, optimization of a linear function of a set of independent variables subject to a set of linear constraints and the requirement that the solution vector have integer components. Special techniques have been developed for the solution of particular CO problems, such as the above example, but there are many important problems (timetabling, scheduling) for which an exact solution in a reasonable time is not always possible. More precisely, there is a large class of CO problems which are known as NP-complete. NP-completeness means that if any algorithm exists which solves a problem in a time which grows as a polynomial in N (a measure of the size of the problem), then all of the set are also soluble in polynomial time. Examples of NP-complete problems include Integer Linear programming, partitioning a set of integers into 2 sets whose sums are equal and the well-known Travelling Salesman problem (TSP) (find the shortest tour or Hamiltonian circuit for a set of coplanar points). The significance of the idea of NP-completeness is that many important problems are NP-complete and therefore, in a sense, equally difficult. In particular, all known algorithms for the TSP run in a time that grows as an exponential in the number of cities in the tour. For this reason, rather than from any

intrinsic interest, the TSP is often used as a benchmark for comparing heuristics, techniques for the approximate solution of CO problems. An important class of heuristics are known collectively as "iterative improvement" methods. The heuristic begins with the system in a particular, often randomly chosen, state. A rearrangement operation (such as 2-OPT for the TSP [1]) is applied randomly until a new state which reduces the objective function is found. The new configuration is adopted and the rearrangements continue until no further improvements can be achieved. Often the heuristic will get 'stuck' in a local, rather than the global, minimum so it is neccessary to use several different initial states and adopt the best final result. In fact, the problem of sticking in a local minimum is not confined to CO, gradient descent and Newton-type methods used in continuous optimization suffer from the same defect.

Simulated Annealing

A more systematic approach to this problem was proposed by S. Kirkpatrick et al.[2] in 1983 based on the Metropolis algorithm. N. Metropolis et al. [3], in 1953, proposed an simple algorithm for the efficient simulation of a collection of atoms in equilibrium at a given temperature. The following exposition of the Metropolis algorithm follows that of S. Geman and D. Geman [4]. Let Ω denote the possible configurations of the system in question; for example ω in Ω could be the molecular positions. If the system is in thermal equilibrium with its surroundings, then the probability of ω is given by

$$P(\omega) = \frac{e^{-\beta E(\omega)}}{\sum_\omega e^{-\beta E(\omega)}}, \qquad \omega \in \Omega \tag{1}$$

where $E(\omega)$ is the energy of the configuration and $\beta = 1/kT$ where k is Boltzmann's constant and T is temperature in degrees Kelvin. The quantities to be calculated are usually ensemble averages of the form

$$\langle Y \rangle = \int_\Omega Y(\omega)\, d\pi(\omega) = \frac{\sum_\omega Y(\omega) e^{-\beta E(\omega)}}{\sum_\omega e^{-\beta E(\omega)}} \tag{2}$$

where Y is some variable of interest. This expression is analytically intractable. In the standard Monte Carlo approach, one restricts the sums above to a sample of ω's drawn uniformly from Ω. This approach fails in the present case due to the exponential factor, as most of the mass of the distribution is concentrated in a very small part of Ω. In other words, for satisfactory accuracy, excessively large samples are needed. The technique introduced in [3] was to chose the samples from P instead of uniformly and then weight the samples uniformly instead of by dP. In other words, one obtains $\omega_1, \ldots, \omega_R$ from P and the ensemble average for Y is approximated by

$$\langle Y \rangle \approx \frac{1}{R} \sum_{r=1}^{R} Y(\omega_r) \tag{3}$$

The sampling algorithm in [3] can be summarized as follows. Given the state of the system at time t, say $X(t)$, one randomly choses another configuration X' and computes the energy change $\Delta E = E(X') - E(X(t))$ and the quantity

$$q = P(X')/P(X(t)) = e^{-\beta \Delta E} \tag{4}$$

If $q > 1$, the move to X' is allowed and $X(t+1) = X'$, while if $q \leq 1$, the transition is made with probability q. Thus, one chooses $0 \leq r \leq 1$ uniformly and sets $X(t+1) = X'$ if $r \leq q$ and $X(t+1) = X(t)$ if $r > q$. Metropolis et al. prove that starting from an arbitrary state, repeated application of this algorithm produces, in the limit of arbitrarily many applications, a sequence of samples from a Boltzmann distribution as stated above. In [2], Kirkpatrick et al. proposed applying the Metropolis algorithm to CO as follows. First select a technique for randomly selecting new states from the current state. For the TSP a widely used technique is Lin's 2-OPT, essentially taking a chain (of a given length) from the current tour and inserting it (possibly reversed in orientation) between two successive points in the tour. Again for TSP, the appropriate 'energy function' is the length of the tour under consideration, for a timetabling problem the energy might be the number of clashes or irreconcilable assignments. The transformation rule (4) is then applied repeatedly until approximate equilibrium is reached at the temperature chosen. The combinatorial system is first 'melted' by being allowed to reach equilibrium at a large value of T. The temperature is then reduced gradually, allowing the system to reach a steady state at each discrete value of T chosen. This decreasing sequence of temperatures is called an annealing schedule by analogy with the slow cooling- annealing- of a melt of a physical substance and the technique itself is called Simulated Annealing (SA) for the same reason. Note that for large values of T (small values of β) new states which increase the energy are likely to be accepted, while for small values of T such uphill moves will be rare. This capacity to escape from local minima is what distinguishes SA from

simple iterative improvement. Physical intuition suggests that cooling must be sufficiently slow to avoid (persisting with the analogy from physics) the formation of a defective crystal or glass, with only locally optimal structures. In fact, it can be proved (see below) that, under certain conditions on the annealing schedule, this procedure will converge to the state corresponding to the minimum of the energy function. In practice, these conditions are not satisfied, but SA still provides 'good' solutions in many cases.

Applications of Simulated Annealing

Much of the early published work on SA has consisted of reports on the results of numerical experimentation and, because of its simplicity and convenience, authors have frequently used TSP to evaluate the technique. Reports on the efficiency of the heuristic vary. C. Skiskim and B. Golden [5] found SA to be inferior to the CCAO procedure and moreover found the performance of SA to be highly sensitive to the details of the annealing schedule. However Skiscim and Golden considered N-city TSP with $N \leq 100$. In a later paper [6] E. Bonomi and J.-L. Lutton found that, for $N \geq 250$, SA outperformed Lin's [1] 2-OPT and the convex hull algorithm. More generally, the technique has proved useful for a wide variety of optimization problems in computer design and other areas; [7,8,9]. I.O. Bohachevsky et al.[10], W. Jeffrey and R. Rosner [11] and others have used Simulated Annealing successfully for optimization of continuous functions of many variables. In work in progress, the present author has applied SA to timetabling and to the Vehicle Routing problem. The latter problem can be posed as a TSP in a natural way with the use of dummy locations corresponding to the vehicular resources. As noted above, SA can be implemented for timetabling using the number of violations of the constraints as the objective function. The major difficulty is choice of a suitable data structure to allow the objective function to be evaluated efficiently. In an influential paper published in 1985, D.H. Ackley, G.E. Hinton and T.E. Sejnowski [12] introduced the idea of the 'Boltzmann machine', a domain-independent learning algorithm which modifies the connection strengths between units of a network in such a way that the whole network developes an internal model which captures the underlying structure of the environment. While space does not permit a full treatment here the following general points may be made. The Boltzmann machine is a neural network or 'parallel distributed processor' as developed initially by J.J. Hopfield [13,14] and others. The machine is composed of elements called units that are connected by symmetric links. A unit is always either on or off, and it adopts these states as a function of the states of the neighbouring units and the weights on its links to them. A unit being on or off is taken to mean that the system either accepts or rejects some elemental hypothesis about the input data (environment). The weight on a link represents a weak constraint between two hypotheses. A variant of the transformation rule (4) is used to modify the state of the individual units of the network so as to bring the network to equilibrium at a given temperature. As always for SA, the temperature is gradually lowered, resulting in (eventually) convergence to a configuration which minimises the objective function. Here the objective function or energy of a configuration is a measure of the extent to which that combination of hypotheses violates the constraints implicit in the input data. The reader is referred to [12,15,16] for further details.

Theoretical Results

The major contribution to the (very small) body of exact results about SA is due to S. Geman and D.Geman [4] (November 1984).In a paper on Bayesian restoration of noisy 2-D images they proved three significant theorems about SA. Here it will suffice to state the three theorems; A, B and C and to discuss their significance. First, some notation is neccessary. (Some changes have been made from that of [4] in the interests of clarity.) Let the state of the system be specified by a vector $x(t)$ with N components x_s. The state-generation process, (without loss of generality), can be required to alter only one component of the state-vector x per update. Let $\{n_t,\ t = 1, 2, \ldots\}$ be the sequence in which the components of x are chosen for updating. Then $\{X(t),\ t = 0, 1, 2, \ldots\}$ is a random process which describes the evolution of the system being studied, where X is a random vector with components X_s, and the evolution $X(t-1) \to X(t)$ of the system is given by

$$P(X_s(t) = x_s, s = 1, \ldots, N) = \Pi(X_{n_t} = x_{n_t} \mid X_s = x_s, s \neq n_t)\, P(X_s(t-1) = x_s, s \neq n_t) \quad (5)$$

where $\Pi = e^{-\beta U} / \sum e^{-\beta U}$ is the Boltzmann factor correspomding to (4) (U corresponds to the energy E to be minimised.) Let the initial configuration of the system be $X(0)$, i.e. the initial distribution $P(X_s(0) = x_s,\ s = 1, \ldots, N)$ is specified for the range of possible values of x_s.

Theorem A (Relaxation) *Assume that for each s, $1 \leq s \leq N$, the sequence $\{n_t,\ t \geq 1\}$ contains s infinitely often. Then for every starting configuration η and every possible state ω,*

$$\lim_{t\to\infty} P(X(t) = \omega \mid X(0) = \eta) = \Pi(\omega) \quad (6)$$

In other words, the distribution of $X(t)$ converges to Π (the Boltzmann or Gibbs distribution) as $t \to \infty$ regardless of $X(0)$. This is essentially a rewording of the result in Metropolis' paper [3].

Some further notation is needed for Theorem B. Rewrite (5) as

$$P(X_s(t) = x_s,\ s = 1, \ldots, N) = \Pi_{T(t)}(X_{n_t} = x_{n_t} \mid X_s = x_s,\ s \neq n_t)\, P(X_{s(t-1)} = x_s,\ s \neq n_t) \quad (7)$$

to indicate the dependence of Π on T, the temperature. The annealing procedure generates a different random process $\{X(t), t = 1, 2, \ldots\}$ for each successive temperature value such that (6) holds. Let $\Omega_0 = \{\omega \in \Omega : U(\omega) = \min_\Omega U(\omega)\}$, that is, the minimum energy configurations of the system and let Π_0 be the uniform distribution on Ω_0. Finally, define $U^* = \max_\Omega U(\omega)$, $U_* = \min_\Omega U(\omega)$ and $\Delta = U^* - U_*$.

Theorem B (Annealing) *Assume that there exists an integer $\tau \geq N$ such that for every $t = 0, 1, 2, \ldots$ we have $\{s_1, \ldots, s_N\} \subset \{n_{t+1}, n_{t+2}, \ldots, n_{t+\tau}\}$. Let $T(t)$ be any decreasing sequence of temperatures for which*

(a) $T(t) \to 0$ *as* $t \to \infty$;

(b) $T(t) \geq N\Delta / \ln(t)$ *for all* $t \geq t_0$ *, for some* $t_0 \geq 2$.

Then for any starting configuration η in Ω and for every ω in Ω,

$$\lim_{t\to\infty} P(X(t) = \omega \mid X(0) = \eta) = \Pi_0(\omega)\,. \quad (8)$$

The first condition merely requires that the update procedure does not slow to an arbitrarily low frequency as the system evolves, and imposes no limitations in practice. Condition (a) is trivially satisfied by any reasonable annealing schedule. However condition (b) is a major problem. For the image restoration problem studied in [4], for example, of the order of e^{40000} updates would be needed to reach $T = 0.5$. Some points can be made. First, condition (b) is a sufficient condition for convergence, and may not be a neccessary one.

(However, the physical process of annealing requires very slow cooling, especially near the freezing point.) Moreover, the modification of SA mentioned in Section 5 below due to H. Szu [17], will, if successful, greatly improve on the performance predicted by Theorem B.

Theorem C (Ergodicity) *As in Theorem B, assume that there exists an integer* $r \geq N$ *such that for every* $t = 0, 1, 2, \ldots$ *we have* $\{s_1, \ldots, s_N\} \subset \{n_{t+1}, n_{t+2}, \ldots, n_{t+r}\}$. *Then for every function* Y *on* Ω *and for every starting configuration* η *in* Ω, *the ergodic hypothesis*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} Y(X(t)) = \int_{\Omega} Y(\omega)\, d\pi(\omega) \tag{9}$$

holds with probability one.

The significance of this result is that time averages rather than phase averages, which are computationally intractable, can be used to compute expected values.

Modifications of the Heuristic

Several modifications to the basic SA heuristic have been suggested. J.W. Greene and K.J. Supowit [18], have proposed a 'rejectionless' form of annealing. They observe that, at low temperatures, the run time is quite high because many candidates are rejected before each move to a different state. This follows for two related reasons. First of all, for T small, the exponential transition probability to states of higher energy is very small. Moreover, at low temperatures, the system is likely to be in a state with only a small proportion of the accessible states having lower energy. Greene and Supowit propose the following alternative strategy. Let x_i, $i = 1, \ldots, N$, be the states accessible (in one move) from the current state of the system. Store $w_i = \min 1, q$, $i = 1, \ldots, N$, where q is given by (4). Then choose state x_i with probability $w_i / \sum w_i$, make the change of state and re-calculate the w_i's. The sequence of states generated by this method is probabilistically equivalent to the corresponding sequence generated by SA, if the repetitions of the current state each time a move is rejected are omitted. This can be demonstrated as follows using the notation of [18]. Let α_{xT} be the probability that SA accepts the

chosen move at temperature T, i.e.

$$\alpha_{xT} = \frac{1}{N} \sum w_i \tag{10}$$

Then the probability that SA makes the move from state x to state x_i (say) after some number of rejections is

$$\sum_{k=0}^{\infty} (1 - \alpha_{xT})^k \frac{1}{N} w_i = \frac{w_i}{N\alpha_{xT}} \tag{11}$$

which is just $w_i / \sum w_j$, the probability of choosing x_i under the rejectionless method. The run time per change of state for rejectionless annealing clearly has a value independent of the acceptance ratio, while for SA the value is proportional to the reciprocal of the acceptance ratio. However the overheads in terms of memory requirements and CPU for the rejectionless method are large, so the method is only useful at very low temperatures. In numerical experiments undertaken by the present author and a student, temperatures sufficiently low to warrant the use of rejectionless annealing were never reached. Another variant on standard SA is due to I.O. Bohachevsky et al.[10]. In a recent paper they propose using a modified form for the transition probability q to states of higher energy (4). For problems where the minimum of the objective function Φ is known to be zero (if the value is non-zero just use Φ less the known minimum value as the energy) they suggest setting $q = \exp(-\beta\Phi^g\Delta\Phi)$ where g is a suitably chosen negative number. The purpose of the modification is to ensure that when close to the minimum, the heuristic is unlikely to move a large distance away. No theoretical analysis of this modified SA is offered but numerical experiments (on optimization of continuous functions of two variables) are quoted which suggest the technique might be useful when the value of the global minimum is known. For the more common situation, where the value of the global minimum is not known, the authors suggest an adaptive approach, starting with an estimate of $\Phi_{\min}$ and modifying it as necessary as the search proceeds. It is not clear how effective this proposal is in practice. Perhaps the most significant modification of SA is that proposed by H. Szu [17] in 1986. As noted in Section 4, the result (8) due to Geman and Geman [4] demands an unacceptably slow cooling rate for guaranteed convergence to the optimal solution. Szu suggests an alternative approach which he calls the 'Cauchy machine', in deference to the Boltzmann machine of Ackley et al. [12]. In standard SA the successive states of the system are generated from a uniform (or, more generally [15], from a Gaussian) distribution. In all cases the distribution is of bounded variance. (The probability of accepting this new state is, of course, given by min{1,q}, where q is given in (4).) In his paper, Szu claims that using the Cauchy distribution, which has unbounded variance, a cooling schedule reciprocal in t, rather than $\ln(t)$, can be used. Unfortunately only a rather unsatisfactory sketch proof is quoted and the reader is referred to an (as yet) unpublished paper for a rigourous derivation [19]. (Some numerical results are produced in support of his assertion.) However, if, as seems likely, Szu's result is valid, the consequences for SA are major. An exponentially faster cooling rate will be possible, making the method far more realistic as a general-purpose optimization technique than previously.

Summary

Simulated Annealing, in various guises, has been in existence for five years and has been applied to a steadily widening range of problems. With developments like those quoted in this review, continued interest in the topic seems assured.

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On The Level

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We present a survey on the notion of the level of a field and its various generalizations. We describe a lot of results that are attractive from an algebraic viewpoint and also highlight the extremely interesting relations between algebra and topology that have been unearthed in the last decade in connection with the level. We hope to persuade the reader that this is an appealing area of mathematics and that it should be a fruitful area for future research. In Section 1, we look at levels of fields, in Section 2, we deal with commutative rings and the link with topology and in Section 3, we look at the non-commutative situation and generalisations of the idea of level.

1 Fields

Let F be a field. F is said to be *formally real* if -1 is not expressible as a sum of squares in F. If F is not formally real we define the *level* of F, denoted $s(F)$, to be the smallest natural number N such that -1 is a sum of N squares in F (We define $s(F) = \infty$ if F is formally real).

The Artin-Schreier theorem [35, p.227] says that a field F is formally real if and only if F admits an ordering (i.e. $s(F) = \infty$ if and only if F admits an ordering).

We look now at levels of some well-known fields

Example 1 $F = \mathbf{R}$, the real numbers, $s(\mathbf{R}) = \infty$.

Example 2 $F = \mathbf{C}$, the complex numbers, $s(\mathbf{C}) = 1$ since $-1 = i^2$ in $\mathbf{C}$.

Example 3 $F = \mathbf{F}_p$, a finite field with p elements, p an odd prime. It is a fairly easy exercise to show $s(\mathbf{F}_p) = 1$ if $p \equiv 1 \pmod 4$ and $s(\mathbf{F}_p) = 2$ if $p \equiv 3 \pmod 4$.

Example 4 $F = \mathbf{Q}_p$, the field of p-adic numbes. Then $s(f) = 1$ if $p \equiv 1 \pmod 4$, $s(F) = 2$ if $p \equiv 3 \pmod 4$. If F is the field of dyadic numbers then $s(F) = 4$. See [35, p.151] for a proof.

Example 5 For F an algebraic number field $s(F) = 1, 2, 4$, or ∞. This is known as Siegel's theorem as it was first proven in [58]. See [35, p.300] for a modern proof.

The notion of level seems to have first explicitly arisen about sixy years ago following the work of Artin and Schreier on formally real fields [4] and Artin's solution of Hilbert's seventeenth problem [3]. It is implicit however in earlier work of Hilbert, Landau and others. Indeed the general problem of representing integers as sums of squares seems to have perpetually engaged the attention of mathematicians, e.g. the work of Diophantos, Fermat, Lagrange, Gauss and numerous others. The German word 'Stufe' was used for the level and this is the reason for the notation $s(F)$.

The suggestion that $s(F)$ is always a power of two if finite was made in [64] and in 1934 H. Kneser [34] proved that $s(F) = 1, 2, 4, 8$ or a multiple of 16. For a period of almost thirty years, no significant advance was made on this although a few authors did examine the level [62, 31, 63, 38]. The major breakthrough came in 1936 when Pfister, inspired by a colloquium lecture at Gottingen by Cassels on sums of squares in rational function fields, succeeded in proving that $s(F)$ must be a power of two if it is finite. (See [51] for Pfister's own account of his discovery). Pfister also succeeded in producing examples of fields of prescribed level 2^k for each positive ineger k. Before this no example of a field of level greater than 4 had been known. Pfister's work appears in [48] and uses quadratic form theory.

It is easy to see that -1 is a sum of squares in the field F if and only if the $(n+1)$-dimensional quadratic form given by the identity matrix is isotropic, i.e. represents zero non-trivially. Thus there is an obvious connection between quadratic form theory and the notion of level of a field. Pfister introduced the notion of a quadratic form being multiplicative and this was the key idea needed to obtain his results on the level of a field. Accounts of his results may be found in [35, Ch.11] and in [56, Ch.2, §10 and Ch.4 §4]. A very quick proof that $s(F)$ is a power of two if finite appears in [56, p.69–72], this proof being a simplification due to Witt of the original proof.

Various authors have obtained results about the levels of specific kinds of fields. Pfister in [50] showed that $s(F) \leq 2^d$ for F non-real and of transcendence degree d over a real closed field. In the realm of number theory the level of cyclotomic fields was studied in [16], [17], [45]. For algebraic number fields in general $s(F) = 1$, 2, or 4 if finite by the theorem of Siegel mentioned earlier. The question of how to distinguish between the cases $s(F) = 1$, $s(F) = 2$, $s(F) = 4$ has been examined in [27, 19, 8]. We quote for example the following theorem of [27].

Theorem *Let F be an algebraic number field. Then $s(F) \leq 2$ if and only if F is totally imaginary and the local degrees at all primes extending the rational prime 2 are even.*

See also [44], [53] for some further results. The question of how the level $s(F)$ is related to other field invariants has been considered. Let $q(F)$ be the cardinality of $\dot{F}/\dot{F}^2$, the group of square classes. Pfister [49] showed that $q(F) \geq 2^{k(k+1)}/2$ where $s = 2^k$ and this was improved by Djokovic [25], using an argument involving graph theory, who showed that for $s > 2$, $q(F) \geq 2^{s+1}/s$ where $s = s(F)$. See also [35, Ch.11] for more information.

2 Commutative Rings

The definition of level is meaningful not just for fields but for any ring with identity element 1. The ring need not be commutative. We deal with commutative case in this section and the non-commutative case in Secion 3. (One could even discuss the level of a non-associative ring with identity but this has not been considered by anyone to the author's knowledge).

Results on $s(R)$ for R the ring of algebraic integers in a p-adic field were obtained by Riehm [55] who showed $s(R) = 1, 2$, or 4 in this case. For R being the ring of algebraic integers in the algebraic number field K results on $s(R)$ were obtained in [26, 46, 47 and 44]. In particular $s(R) \leq 4$ when $s(K) < \infty$ is proved in [46], and in [26] it is proved that $s(R) = 1$ if $s(K) = 1$, $s(R) \leq 3$ if $s(K) \leq 2$ and $s(R) \leq 4$ if $s(K) \leq 4$, this theorem being attributed to M. Kneser. For further information see the above references.

It is easy to see that the level of commutative ring need not always be a power of two.

Example $R = \{0, 1, 2, 3\}$ with addition and multiplication modulo four. Then $s(R) = 3$ because $-1 = 3$ and 1 is the only non-zero square in R.

Knebusch [32] proved that $s(R)$ is a power of two when R is a local ring in which 2 is a unit. Baeza followed this up by proving the same result for semi-local rings with 2 a unit and had more results on levels of rings in [5], [7, app.1], and [6]. In particular he proved in [6] that, for a Dedekind domain R with field of fractions F, $s(F) \leq s(R) \leq 1 + s(F)$.

A major landmark in the theory of levels occurred in 1979 when Dai, Lam and Peng, [24] proved the following:-

Theorem *Any positive integer may occur as the level of a commutative ring.*

The sensational feature of their work was that they proved this theorem by appealing to a theorem from topology. Their proof goes as follows:-

Let
$$R = \frac{\mathbb{R}[x_1,\ldots,x_n]}{(1+x_1^2+x_2^2+\cdots+x_n^2)}$$
i.e. the quotient of the Polynomial ring $\mathbb{R}[x_1,\cdots,x_n]$ by the ideal generated by $1+x_1^2+x_2^2+\cdots+x_n^2$. Clearly $s(R)\le n$ and the problem is to show $s(R)<n$ is impossible. Suppose -1 is a sum of $n-1$ squares in R. Then there exist polynomials $p_j(x_1,x_2,\cdots,x_n), j=1,2,\cdots,n-1$ and $q(x_1,x_2,\cdots,x_n)$ such that
$$-1=\sum_{j=1}^{n-1}p_j^2+q\left(1+\sum j=1^{n-1}x_j^2\right)$$
The trick is to replace $x=(x_1,x_2,\ldots,x_n)$ by $ix=(ix_1,\ldots,ix_n)$ where $i^2=-1$. Then we may write $p_j(ix)=r_j(x)+is_j(x)$, r_j and s_j being real polynomials, r_j being even, i.e. $r_j(-x)=r_j(x)$, and s_j being odd, i.e. $s_j(-x)=-s_j(x)$.

Now define a map $f:S^{n-1}\longrightarrow\mathbb{R}^{n-1}$ by
$$f(x)=(s_1(x),s_2(x),\ldots,s_{n-1}(x)) \text{ for each } x\in S^{n-1}$$
Since f is continuous we may apply the Borsuk-Ulam theorem from topology [59,p.266] which says that there must exist a pair of antipodal points of S^{n-1} mapped to the same element of $\mathbb{R}^{n-1}$ i.e. $f(z)=f(-z)$ for some $z\in S^{n-1}$. But $f(-x)=-f(x)$ for all x because each s_j is odd and thus $f(z)=0$ i.e. $s_j(z)=0$ for each j. This implies that $-1=\sum_{j=1}^{n-1}r_j(z)^2$, i.e. -1 is a sum of squares in $\mathbb{R}$, completing the proof by contradiction.

After the Dai-Lam-Peng paper had appeared algebraic Borsuk-Ulam theorems were proven by Arason-Pfister [2] and also by Knebusch [33]. The theorem of [2] goes as follows:-

Theorem *Let $f_1,f_2,\ldots,f_{n-1}$ be a set of polynomials in $x=(x_1,x_2,\ldots,x_n)$ with coefficients in a real closed field F. Assume the f_j are odd, i.e. $f_j(-x)=-f_j(x)$. Then there exists $z\in S^{n-1}$ for which $f_j(z)=0$ for all j.*

It is easy to show that the above theorem is equivalent to the statement that given a set of polynomials $f_1,\ldots,f_{n-1}$ in x there exists $z\in S^{n-1}$ with $f_j(-z)=f_j(z)$ for each j. Write each f_j as an even plus an odd polynomial!

This algebraic Borsuk-Ulam theorem for polynomials in fact will yield the full Borsuk-Ulam for continuous functions by using the Weierstrass approximation theorem and the compactness of S^{n-1}.

Proof We briefly outline the proof. Introduce an extra indeterminante x_0 and multiply each monomial in f_j by a suitable power of x_0 so as to make a homogeneous polynomial $\tilde{f}_j$. Replace x_0^2 by $x_1^2+x_2^2+\cdots+x_{n-1}^2$ (x_0 appears in even powers because f_j is odd) and obtain f_j which are homogeneous polynomials of odd degree in $x_1,x_2,\ldots,x_{n-1}$. Now aplying a theorem of Lang [36] these polynomials must have a common zero in F^n which we may take to be in S^{n-1}. (Dividing by $\sqrt{\sum x_i^2}$ is all right as they are homogeneous!)

Dai and Lam [23] investigated in much greater detail the links with topology that had been forged in [24]. They discovered that the level in algebra is closely related to notions in topology that had been considered earlier by C.T. Yang [65, 66] and by Conner and Floyd [20, 21]. We describe this now.

Let $(X,-)$ be a topological space equipped with an involution $-$, i.e. a continuous map $X\longrightarrow X$, $x\rightarrow\overline{x}$ of a period two (so that $\overline{\overline{x}}=x$)

Example 1 $X=S^n$, $-$: the antipodal map.

Example 2 $X=\mathbb{C}$, $-$: complex conjugation.

Example 3 $X=$ the Stiefel manifold $V_{n,m}$ of orthonormal m-frames in $\mathbb{R}^n$, with involution ε_r givenby
$$\varepsilon_r(v_1,\ldots v_r,v_{r+1},\ldots v_m)=(v_1,\ldots,v_r,-v_{r+1},\ldots,-v_m)$$
An equivariant map between $(X,-)$ and $(Y,-)$ is a continuous map $f: X\longrightarrow Y$ such that $f(\overline{x})=\overline{f(x)}$ for all $x\in X$. The *level* of the space $(X,-)$ is then denoted $s(X,-)$ and is defined by
$$s(X,-)=\inf\{n:\text{there exists an equivariant map from }(X,-)\text{ to }(S^{n-1},-)\}$$
Essentially the same invariant as $s(X,-)$ had been studied earlier in [20, 21] where it was called the co-index.

The link with algebra is obtained by associating to $(X,-)$ the ring of all equivariant maps from $(V,-)$ to $(\mathbb{C},-)$. This ring is denoted A_X.

Theorem (Dai, Lam [23]) $s(X,-) = s(A_X)$

They also define the *colevel*

$s'(X,-) = \sup\{n :$ there exists an equivariant map from$(S^{n-1},-)$ to $(X,-)\}$

Motivated by this topological notion of co-level one may define for any real algebra A an algebraic *co-level*

$s'(A) = sup\{n :$ there exists a real algebra homomorphism from A to $A_{S^{n-1}}\}$

It is easy to see that, for any $(X,-)$, $s'(X,-) \leq s'(A_X)$. Dai and Lam proved [23] that if X is a real affine variety then $s'(X,-) = s'(A_X)$.

Another interesting and related notion examined in [23] is that of the *sublevel* of a commutative ring R, denoted $\sigma(R)$. We say $\sigma(R) = n$ if $0 = a_1^2 + a_2^2 + \cdots + a_{n+1}^2$ for elements $a_1, a_2, \ldots, a_{n+1}$ such that the ideal generated by these $(n+1)$ elements is the whole of R and N is the least integer for which this property holds. One notes that $\sigma(F) = s(F)$ for any field F and that $\sigma(R) \leq s(R)$ for any R. The simplest example of a ring R where $\sigma(R) \neq s(R)$ seems to be

$$R = \frac{\mathrm{Q}[x,y]}{(1+x^2+2y^2)}$$

for which it can be shown that $\sigma(R) = 2$ but $s(R) = 3$. See [23] and [15] for proof.

If $s(R) = 1, 2, 4$, or 8 it can be shown that $\sigma(R) = s(R)$ by using the 2-square, 4-square or 8-square identities [29, p.417].

For a commutative ring in which 2 is a unit it is not too hard to show [23] that $s(R) = \sigma(R)$ or $1 + \sigma(R)$. The following natural question was posed and answered in [23].

Which pairs (n,n) and $(n,n+1)$ occur as $(\sigma(R), s(R))$ for some R? They showed that (n,n) occurs for all n and that $(n,n+1)$ occurs for all $n = 1, 2, 4$ or 8. They exhibited examples for all these cases. For $n(n,n+1)$ their examples are the rings

$$\frac{\mathbf{R}[x_1, x_2, \cdots, x_{n+1}, y_1, y_2, \cdots, y_{n+1}]}{(1 - \sum x_i^2, 1 + \sum y_i^2, \sum x_i y_i)}$$

To prove $\sigma(R) = n$ and $s(R) = n+1$ involves relating $\sigma(R)$ and $s(R)$ to the level of certain Stiefel manifolds and calculation of the level of these

appeals to non-trivial topological results. (In particular Adams' result on the non-existence of elements of Hopf invariant one). We refer the reader to [23] for the details. There are many more interesting connections with topology in [23], in particular using results on equivariant maps into Stiefel manifolds. For example Adams' theorem on vector fields on spheres may be used to show that for any commutative ring R, if the form $n \times < 1 >$ over R represents -1 then in fact the form $n \times < 1 >$ contains $\rho(n) \times < -1 >$ as an orthogonal summand. (Here $\rho(n)$ is the Hurwitz-Radon number [35, p.131]).

We should also mention one question raised in [23] and still unsolved at present, namely the Level Conjecture. Let C be a commutative ring and

$$R = \frac{\mathrm{C}[x_1, x_2, + \cdots + x_n]}{(1 + x_1^2 + \ldots + x_n^2)}$$

The Level Conjecture is that $s(R) = n$. For $C = \mathbf{R}$ we have described the proof and Arason-Pfister [2] have proved it when C is any field. It is not clear what technique to use for an arbitrary commutative ring C.

Recently much progress has been made on the study of levels in connection with real algebraic geometry. The following lemma is a starting point for some of this theory.

Lemma *Let R be a commutative ring with 1. Then $s(R) < \infty$ if and only if $s(F(R/\wp)) < \infty$ for all prime ideals $\wp$ of R, $F(R/\wp)$ denoting the field of fractions of the integral domain $R/\wp$.*

Proof See [18], [12] or [22] where it was first observed. See [18] for how this leads to the Real Nullstellensatz and Positivstellensatz in real algebraic geometry.

When R is the co-ordinate ring of an affine variety V without any real points Mahé has succeeded in finding a bound for $s(R)$ in terms of the Krull dimension of R (One may show easily that V has no real points if and only if $s(R) < \infty$).

Theorem *Let F be a real closed field and A an $F-$algebra of finite type with Krull dimension d, specA having no real points. Then $s(A) \leq d - 1 + 2^{d+1}$.*

Proof See [43]. This theorem answers question 11.3 posed in [23].

We finish this section by pointing out that levels are only one aspect of the general study of sums of squares. Throughout the history of mathematics sums of squares have been a topic of fascination and curiosity. Some general references are [28, 61]. One particular problem is that of the *Pythagoras number* $p(R)$ for a commutative ring R. We define $p(R)$ to be the least integer n such that every sum of squares in R is a sum of at most n squares. The determination of $p(R)$ is generally a very difficult problem. See [14, 15], for further information and references. One may also examine k-th powers instead of squares and can generalize the level by asking for the least n such that -1 is a sum of n k-th powers. (k should be even as it is trivial for odd k). See [10, 9] for information on this for fields, also [30] for rings.

3 Non-Commutative Rings

There is very little in the literature about levels or sums of squares in the non-commutative situation. The following theorems were proved recently.

Theorem (Leep, Shapiro and Wadsworth) *Let D be a division algebra finite dimensional over its centre F. Then the following three statements are equivalent:*

(i) 0 is a non-trivial sum of squares in D;

(ii) -1 is a sum of squares in D;

(iii) each element of D is a sum of squares in D.

Proof See [37]. Note that if D is a field this theorem is an easy exercise.

A quadratic form q over a field F is *weakly isotropic* if, for some n, the orthogonal sum of n copies of q is isotropic.

Theorem *Let D be a division algebra finite dimensional over its centre F. Then 0 is a non-trivial sum of squares in d if and only if the trace form of D is weakly isotropic.*

(Note: the trace form is the map $q : D \longrightarrow F$, $q(x) = \text{tr } x$, tr being the reduced trace [56, p.296].

Proof See [39].

It follows that $s(D) < \infty$ if and only if the trace form of D is weakly isotropic for D as in the above theorems. In [40] we examined the case of D being a quaternion division algebra and obtained the following results.

Theorem *There are quaternion division algebras D wih $s(D) = 2^k$ for any k and with $s(D) = 2^k + 1$ for any k.*

(It is an open question whether or not other integer values can occur as $s(D)$ for quaternion algebras.) The examples with level 2^k and $2^k + 1$ are described as follows: Let $F = K((t))$, the Laurent series field in one variable t and let $K = \mathbb{R}(x_1, x_2, \ldots, x_n)$, the rational function field in $x_1, x_2, \ldots, x_n$. Let $D = \left(\frac{a,t}{F}\right)$ where $a = \sum_{i=1}^n x_i^2$, i.e. D is the quaternion algebra defined by $i^2 = a, j^2 = t$ etc. For $n = 2^k+1$ it is shown that $s(D) = n$. For $n = 2^k$ we use $F = K(t)$, the rational function field, K as above, but let $D = \left(\frac{-t, t-a}{F}\right)$ and then it turns out that $d(D) = n$. Our techniques make use of Pfister's results on products of sums of squares.

One may also consider sublevels for non-commutative rings and a few results appear in [41], mainly for quaternion algebras.

From one point of view it may be argued that the appropriate generalization of sums of squares to the non-commutative case is sums of products of squares. For example Szele [60] proved the following generalization of the Artin-Schreier theorem.

Theorem *Let D be any skewfield. Then D admits an ordering if and only if -1 is not a sum of products of squares in D.*

This suggests one possible generalization of level to what we will call the *product level* and denote $s_\pi(R)$ for any ring R. The *product level* $s_\pi(R)$ is the least integer n such that -1 is a sum of n products of squares in r. Define $s_\pi(R) = \infty$ if -1 is not a sum of products of squares in R. Szele's theorem thus may be rephrased as $s_\pi(D) = \infty$ if and only if D admits an ordering.

Not also that Albert [1] proved that an ordered skew-field must be infinite-dimensional over its centre and thus $s_\pi(d) < \infty$ for finite dimensional algebras.

The only result in the literature on s_π is the following due to Scharlau and Tschimmel.

Theorem *Every positive integer can occur as $s_\pi(d)$ for some skewfield D.*

Proof See [57].

The examples prduced in [57] are all infinite-dimensional over their centre so it would still seem open as to whether or not $s_\pi(D)$ can take every positive integer value for skew-fields finite over their centre.

We now make a few elementary observations about $s_\pi(D)$ for skewfields D.

(1) It is possible to have $s(D) = \infty$ but $s_\pi(D) = 1$. An example of this appears in [37]. Let $D = \left(\frac{x,y}{F}\right)$ where $F = \mathbb{Q}(x,y)$ the field of rational functions in two variable x, y over the field $\mathbb{Q}$ of rational numbers. Then $s(D) = \infty$ since the trace form of D is the form $< 1, x, y, -xy >$ which is not weakly isotropic over F. However $s_\pi(D) = 1$ since this is true for any quaternion algebra as $-1 = i^2 j^2 ((ij)^{-1})^2$.

(2) Any commutator $[x, y] = xyx^{-1}y^{-1}$ in D is a product of squares since $[x,y] = x^2(x^{-1}y)^2(y^{-1})^2$. Thus whenever D contains a pair of elements which anti-commute then $s_\pi(D) = 1$. In particular $s_\pi(D) = 1$ for any quaternion algebra D.

(3) Let D be finite-dimensional over its centre F and let N be the reduced norm map D to F. See [56, p.296] for a definition of N. If D is odd-dimensional over F then $s_\pi(D) > 1$ provided -1 is not a square in F.

Proof $N(-1) = -1$ for n odd whereas if $-1 = x_1^2 x_2^2 \ldots x_n^2$ then $N(-1) = N(x_1)^2 N(x_2)^2 \ldots N(x_n)^2$ is a square in F. Contradiction.

Theorem *If D is even-dimensional over F then $s_\pi(D) = 1$ if the reduced Whitehead group $SK_1(D) = 1$*

Proof $SK_1(D) = \{x \in D : N(x) = 1\}/[D, D]$ so if $SK_1(D) = 1$ every element of norm 1 is a product of commutators and hence a product of squares. $N(-1) = 1$ for even-dimensional D so that $s_\pi(D) = 1$.

It seems quite likely that $s_\pi(D) = 1$ always for D even-dimensional over its centre but we cannot prove this. [1]

(4) Let $\dot{D}$ be the multiplicative group of non-zero elements of D and $\dot{P}$ the normal subgroup of all non-zero products of squares. Then $s_\pi(D) \leq |\dot{D}/\dot{P}|$, the index of $\dot{P}$ in $\dot{D}$.

[1] A. Wadsworth has just proved this!

Proof Let $-1 = \sum_{i=1}^{n} p_i$, each $p_i \in \dot{P}$, n minimal. Let $b_k = \sum_{i=1}^{k} p_i$ for each $k = 1, 2, \ldots, n$. Then $b_1, b_2, \ldots, b_n$ give distinct cosets of $\dot{P}$ in $\dot{D}$ since n is minimal. Thus $s_\pi(D) \leq |\dot{D}/\dot{P}|$.

We describe now another possible generalization of level which can be defined in both the commutative and non-commutative cases.

Let R be a ring with an identity and equipped with an evolution, i.e. an anti-automorphism of period two. We use the symbol to denote an involution. A *hermitian square* in R is an element of the form $\overline{x}x$ for some $x \in R$.

We define the *hermitian level* of the ring with involution $(R, -)$ to be the least integer n for which -1 is a sum of n hermitian squares in $(R, -)$. We write $s_h(R, -)$ for the hermitian level and define $s_h(R, -) = \infty$ if -1 is not expressible as a sum of hermitian squares. Note that $s_h(R, -)$ depends on the involution $-$ and not just the ring R. The general idea for obtaining results on $s_h(R, -)$ is to use hermitian form theory in the same way that quadratic form theory is used to obtain results on the usual notion of level. This is done in [42] and we summarise some of the main results obtained there.

(1) Any positive integer n may occur as $s_h(R, -)$ for a ring with non-trivial involution. Specifically $s_h(RC_3, -) = n$ where R is a ring with $s(R) = n$, such as described in Section 2, C_3 is the cyclic group of order three and $-$ is the involution on the group ring RC_3 induced by $\overline{g} = g^{-1}$.

(2) Let $(R, -)$ be either a field with involution or a quaternion division algebra with standard involution. Then $s_h(R, -)$ is a power of two if it is finite.

(3) Let $D = \left(\frac{a,b}{F}\right)$, the quaternion division algebra with non-standard involution $\hat{}$ defined by $\hat{\imath} = -i, \hat{\jmath} = j$. If F is real closed or p-adic then $s_h(D, \hat{}) = 1$. If F is a number field then $s_h(D, \hat{}) = 1, 2,$ or ∞.

(4) It is an open question as to whether or not $s_h(D, \hat{})$ must always be a power of two for quaternion division algebra in general. We show in [42] that every power of two occurs as $s_h(D, \hat{})$ and also that integers of the form $2^k - 1$ cannot occur but we have no other results on this question.

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A Sociological Question

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Write on John H. White's theory of "open" and "closed" Catholicism, in the context of religion in modern Irish society.

— from a Maynooth BA exam paper.

Let OP be the set of all possible opinions. When endowed with Archdeacon Wellbeloved's aggiornamento topology (the topology of substantial agreement on the broad fundamentals of the question), OP becomes a completely regular connected Hausdorff topological space. Regrettably, OP satisfies neither the first nor the second axiom of countability, and hence is non-metrizable, but then you can't have everything. The space OP contains non-contractible loopy sets of opinions, and hence is not simply-connected. The problems this poses may sometimes be overcome by passing to the universal covering space, the space of all idee-fixed homotopy classes of circular arguments, also known as full socio-loopy space.

A *person* is a set-valued function p, defined on the set $\mathbb{R}$ of all real numbers, with values in the power set of OP. The majority of persons ordinarily encountered have the additional property that $p(t)$ is empty before an initial conception-time, depending on the person (depending on some other persons, too, who enjoy it a lot more). It is also usually found that $p(t)$ remains constant once t exceeds about 15 years after conception-time. The technical term for this is that $p(t)$ has been *set in concrete*.

Let N denote the set of propositions contained in the Nicene Creed.
Let A denote the set of propositions contained in the Apostle's Creed.
Let I denote the singleton: { The Pope is tops }.

Definition A person p is a *catholic* at time t if and only if $p(t)$ contains the union of N, A, and I.

Evidently, a catholic is open at time t if it holds a neighbourhood of each of its opinions. It is closed if it holds an opinion x whenever it holds opinions arbitrarily close to x.

Theorem 1 *All open-closed catholics are loopy.*

Proof The space of opinions is connected, because some people clearly find it possible to agree with everybody. Thus an open-closed catholic must hold all opinions, including inconsistent pairs of opinions. QED

Thus, at any given time, a loopless catholic may be open or closed, or neither, but not both.

Lemma 1 *The Pope is a closed catholic.*

Proof That the Pope is a catholic is well-known (cf. Mahoney, Acta Apost. Sedis, 1(33)1) . If an opinion y is close to an opinion x, but not equal to x, then the Pope holds at most one of them, by the Infallibilitatsatz. Thus the Pope's opinions are discreet. Oops, discrete. QED

Corollary 1 *If a catholic has trivial fundamental group, then it is closed.*

Proof Let $c(t)$ be a homotopically-trivial catholic. Then $c(t)$ is loop-free, and hence agrees with the Pope, period. The result then follows at once from the lemma. QED

It should be noted here that Lefebvre, in his article on the application of sheaf-theory to the exegesis of the Reaper-parable, has established that coherent analytic catholics are bl—dy fascists. This result is, of course, a simple consequence of the above corollary, as is seen by referring to the pontifical exact spectral sequence.

This leaves us with only one class of catholics to consider: the loopy, non-open-closed class. Passing to full socio-loopy space, we observe that the Pope remains discreet , when lifted, because the covering map is a local homeomorphism. All catholics become loopless when raised, and hence closed. We may not conclude from this that the unlifted catholics are closed, or even Suslin-analytic, since the spaces involved are not Polish. We have proved:

Theorem 2 *All uplifted catholics are closed. Some loopy ones are also open.*

This brings us to cardinals. A *cardinal* is a catholic who is discrete, compact, totally-ordered and well-ordered by inclusion, and humble beyond belief. Thus, if x is a cardinal, then it holds each of the opinions of all its predecessors.

Theorem 3 *There are no inaccessible cardinals.*

Proof Since cardinals are discrete and compact, they are automatically finite, hence do not go beyond the opinions of their predecessors, and thus they are accessible. QED

The results may be applied to modern Irish society in a subsequent paper, unless the author is silenced.

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ABOUT TeX

Charles Nash Richard M. Timoney

Why TeX?

Perhaps every mathematician is aware by now of the existence of TeX, a system for mathematical word processing and typesetting designed by the respected mathematician D. Knuth. This system has attracted some dissension among mathematicians, but it is rapidly becoming the standard system for preparing mathematical papers and documents because it is recognised even by its critics as the system which produces the best available output.

The word TeX is a shibboleth for the initiated of the art of mathematical word processing: TeX is pronounced "tech" with the ch imitating that found in loch. As Knuth explains this is because TeX represents the Greek root $\tau\epsilon\chi$.

The reason TeX produces such satisfactory output is that Knuth has put quite an enormous amount of effort into encapsulating within TeX the wisdom accumulated by professional typesetters over the centuries. In this sense TeX is almost an expert system; it makes decisions of its own — how far above the line a superscript should be, how small it should be, whether the limits of integration should be beside the integral sign on above and below, etc. It incorporates this kind of knowledge in respect of ordinary text also — spacing between letters or words, sensible places to hyphenate English words where necessary, for example. It does allow you to override the decisions it makes if you really want to, but the reason the finished product is so excellent is largely because this kind of detail is built into the system.

It is clear (to us) that the era of books or conference proceedings which have been reproduced directly from typewritten copy is nearly over. Soon papers typed on a golf-ball typewriter will be as unacceptable as a typescript with handwritten symbols is today. Just as twenty years ago mathematics departments adopted the golf-ball typewriter and coped with the consequential need to train secretaries in their use, they must now inevitably find the resources needed to implement TeX.

As some of you may know TeX runs on both PC's and mainframes and as it is a very powerful piece of software one would be forgiven for expecting it to be very expensive to buy. Happily this is not so: on mainframes the TeX

programme is in the public domain which means its cost is nominal being about $50 from the AMS. On PC's there are packages available which run TeX and these are a little more expensive. For example a commonly used one is called PCTeX and costs $248. This price should come down though in the future. The main reason for TeX not costing thousands of dollars is that Don Knuth has very generously given the programme to the public domain. Not only this, he also donates all his royalties from the manuals he has written for TeX to the AMS TeX users group: Tugboat.

It is AMS policy to support TeX and Maths Reviews is completely typeset in TeX as may be seen by looking for the appropriate small print on the inside cover. Thus although TeX is not perfect—it is still considerably easier to fill in a form using a typewriter than to try and use TeX—it does seem to be much better than any alternatives [1] and this is particularly true for text containing lots of mathematical symbols.

We think most mathematicians roughly understand the capabilities of a golf-ball typewriter and this influences their expectations of the result when something is typed on such a typewriter. With TeX, we think it will also be necessary for mathematicians to have some appreciation of what can be done, what is easy, what is difficult and what should not be asked of it. Some changes of attitude are required; documents prepared using TeX have more or less the same constraints as one is used to in a journal. Somehow we acquire (to a greater or lesser extent) an appreciation of good practice in preparing an article for submission to a journal. Long formulae should be displayed rather than embedded in the text; underlining is almost never used in printed material — bold face or italic type is used instead; symbols x, y, $f(x)$, etc are set in italic type but sin, sup, inf, etc. are not.

This rather superficial change of attitude by the mathematicians content to leave the details to someone else is not often considered. A lot of weight is frequently placed on the difficulty of learning how to use TeX. It is true that TeX involves creating first an abstract representation of the intended output document. In this sense TeX is a kind of programming language, but this feature of it makes it attractive to many mathematicians. As against that, simple documents are simple. A text document with no formulae is just typed in as it is with the understanding that TeX will decide how much to put on each line and that one begins a new paragraph by leaving a blank line. As one would expect, it takes little more know-how to centre a line or to put

[1]This view is broadly similar to those expressed by Abikoff [1] and Palais [2].

something in larger or bold-face type. To start a formula (the equivalent of changing to the symbol golf-ball) you type $ (or perhaps $$ for a displayed formula) and you end the formula with a matching $. Inside a formula a simple subscript such as x_i is indicated by an underscore `x_i`, a superscript or exponent as in x^2 by a carat `x^2`, the greek letter α by `\alpha`, and so on. As one would expect, compound subscripts are marginally harder, matrices are a little harder again, etc. Once typed into the computer — and this can be done with almost any text editor and on almost any computer (not necessarily one that can run the TEX program itself) — one runs the result through the TEX program and the end product can be printed (or viewed on the computer screen if you have appropriate equipment).

We admit all this takes a little getting used to and it helps a lot to have someone down the hall who doesn't mind answering questions that begin "How do you do ...?". We don't think it is an insurmountable difficulty for anyone to learn, but some effort is required.

Soon perhaps we will all be preparing our own papers, exams and hand-outs in TEX on the micro in the office. There is a growing school of thought among those who can type reasonably fast that it is quicker to type in the TEX input than to write out a draft which is sufficiently tidy for a secretary to read. In the shorter term however, it is inevitable that secretarial work becomes computerised and, for the secretaries in the mathematics department TEX is the right system to have.

We present below our own experiences with TEX. We know that TEX has also been adopted at other institutions in Ireland. Interesting developments are taking place at UCD, where Wayne Sullivan is writing a TEX previewer and a Laserwriter driver, both of which promise to be much faster than those currently available. Those interested in getting a preliminary version of his previewer should contact WSULIVAN@UCD.IRLEARN (HEANET) (WSULIVAN@IRLEARN.BITNET).

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TEX at TCD

It is probably accurate to say that Timothy Murphy in TCD was the first Irish person on the TEX scene and he set up experimental TEX outputting programs (referred to as printer 'drivers') at TCD at a very early date. Despite this, due to lack of enthusiasm on the part of the TCD computer laboratory and lack of resources under the control of the school of mathematics, it was only recently that a satisfactory system for TEX has become available to users in the mathematics departments. For some years there was a lone Toshiba printer attached to the grossly overloaded DEC20 mainframe which was capable of printing TEX output. The whole system was slow and painful to use and it was necessary to go to the computer laboratory and physically adjust the printer for TEX each time one used it.

For almost a year we have had an Apple Laserwriter attached to an Ergo microcomputer (that's an IBM-PC clone with a 20MB hard disc, 640K of memory, a Hercules graphics screen and TEX software from PCTEX). This is linked to the departmental minicomputer system (via software known as 'Kermit') so that TEX input files can be typed in on any of a number of terminals in the department. The system is quite satisfactory although it is still necessary to use the Ergo for finally printing (or even error-checking) the input. We dream of a high speed network in the department with various processors attached which would allow (among other things, of course) anyone at any terminal to check a TEX input file for errors and send it off to be printed. The PC linked directly to the printer might then only be used for confidential documents and examination papers.

More than half the academic staff in the school of mathematics have experience of using TEX personally and the facility is quite heavily used. Regrettably the one secretary we had who was competent at producing mathematical papers in TEX has left but we hope to remedy this soon.

How fast is TEX on our present set up? This is not too easy to answer accurately. I took a short paper with a fairly high density of mathematical formulae — integrals, partial derivatives, subscripts, *etc.* It took about a minute to run through the TEX program (this step might need to be repeated if errors crop up). Printing it out in 12 point type size (over 3.5 pages) took about 7 minutes. The average time per page would be shorter for a longer document (under a minute per page). Some of the time is occupied by the computer translating the DVI (device independent) output produced by the TEX program into a different format (about 2.5 minutes in the example) and some by initial overhead so that it took over 5 minutes for the first page to appear. Thereafter the printer prints at about 2 pages per minute. The print quality is high and the printer is not noisy.

There is also a screen preview facility which allows one to see the general layout and to understand errors that have shown up in running the TEX program. This is not quite instantaneous but is much faster than printing as a way of looking through for things that are not coming out as you wanted. It has the drawback that it takes a while to start up (perhaps 30 seconds), it only allows you to see a portion of the page at a time and it takes a few seconds to move the 'window' around the page or to the next page. This program requires better graphics (Hercules or EGA) than the minimum CGA standard. The rest of the software runs happily without good graphics. Preview is reputed to be noticably faster on a PC-AT.

On the subject of macro packages for TEX, we have tended towards LATEX rather than the plain or AMS packages. For example we have LATEX style files and blank documents for letters, memos and exam papers which we are beginning to use. I should say that some of us do use plain TEX and all three packages are available. *RMT*

TEX at Maynooth

We have had TEX running and printing at Maynooth since the summer of 1983. There are now many TEX users here and they are by no means all members of the mathematics departments. For example they may be found among the historians, linguists and sociologists; also the computer laboratory use TEX to produce its monthly newsletter while the library uses it for special high quality notices. One can infer from the above remarks that, for non-mathematical prose, TEX is rather easy to use. In addition considerable resources are needed here to meet the demand created by all these TEX users.

In fact the most critical resources for TEX users tend to be the printers. The two mainframes here (a vax and a microvax) support four printers for TEX printing. Two of these are medium resolution Toshiba dot matrix printers. The other two are high resolution DEC LN03 laser printers. All four printers are permanently switched on and offer queueing facilities: thus a TEX job can be submitted to any of these devices from any terminal, then the job either starts immediately if the relevant queue is empty, or awaits its turn if the printer is already in use. The Toshiba printers are rather slow taking three minutes or so to print an average A4 page. However they are fine for

drafting and for when one is not in a hurry. The laser printers are extremely fast printing at a rate of six or seven A4 pages a minute even for the most heavily symbol-laden mathematical text. A word of caution: not all laser printers reach these speeds, particularly if, as is *not* the case for the LN03's, they use the page description language called Postcript. The siting of printers around the campus is an important topic and one of obvious interest to mathematicians.

The mathematics departments have in their corridor a room in which one of the Toshiba devices referred to above is located. Also in this room we have an IBM PC clone known as a Prompt to which we have connected an Apple Laserwriter. This latter arrangement frees us from being dependent on the mainframes whose queues can become rather long and slow when these mainframes are heavily loaded. The PC can double as a terminal to the mainframes so that we may use Kermit to send files back and forth between the two sorts of machine. The Apple Laserwriter is not quite as fast as the LN03 particularly for mathematical text using many symbols per square cm. I have referred in passing above to one of the reasons for this. Nevertheless PC-based TEX systems are likely to become ever more popular in the future. This is because the large fall in price of both PC's and laser printers of recent years makes their purchase an option for indivdual departments. As well as this such a system operates independently of a potentialy heavily loaded mainframe but can be connected to it using Kermit if it suits one. Many mathematicians now boast PC's either at home or in their offices and this allows them both to make full use of all the college resources and to take their work away with them on a diskette.

Lastly electronic mail, both national and international, is here to stay and a TEX file is an obvious medium in which to send a mathematical article. The author has actually published an article in a conference proceedings abroad which was submitted electronically at the request of the conference organisers — they had some industrial action in their postal service at the time of the conference. In the case of journals, the requirement of a uniform appearance, and the need not to penalise would-be authors whose institutions do not provide the relevant mathematical word processing facilities, makes routine electronic submission look a little further in the future. *CN*

Axiomatic Method and Independence Results[1]

Radoslav Dimitrić

Ever since the discovery of non-Euclidean geometries, mathematicians were interested in formal methods and axiomatization of mathematical theories. It became apparent that ever present Euclidean geometry was not the only true geometrical reality, but that it could rather be substituted by other geometries, equally good and interesting on their own.

I will not exaggerate if I say that modern mathematics (by that I mean this century's mathematics) has been dominated by the use of formal i.e. axiomatic method. The aim of this article is to give a brief survey of axiomatic method with a few concrete applications.

Foundations of Geometry

There is little doubt as to whether the thirteen books of Euclid's "Elements" were the most valuable and influential scientific books of all time, if for nothing else but for the length of time during which they maintained their importance and influence in Mathematics, research and education. For over 2000 years the "Elements" were the standard of mathematical rigour, clarity and "absolute truth".

As it is always the case in scientific progress, however, there has never been room for contentment with any scientific achievement and this applied even to so "perfect" a work as Euclid's. Ever since the appearance of "Elements" there were questions about independence and consistency of Euclid's postulates; could any of the postulates (a more customary modern word for them is axioms) be derived as a theorem from the rest of the postulates? A special attention was paid to the famous fifth postulate:

[1]A lecture of a similar content was delivered by the author at the conference "Groups in Galway", Ireland, 09 - 10 May 1986.

Axiom E *For any plane π , any line $l \subset \pi$ and any point $A \in \pi \setminus l$ there exists at most one line k containing the point A and not intersecting the line l .*

This postulate was shown to be equivalent (given all other axioms) to the statement that there exists one rectangle or that the sum of the angles of a triangle equals to two right angles etc. Among the most penetrating mathematicians working on the subject were G. Saccheri (1667 - 1733) and J.H. Lambert (1728 - 1777) who have developed geometry arising from axioms without fifth postulate or its equivalents.

In the year 1829 the foundations of a great part of the 20th century mathematics (and we can safely say of the 20th century physics as well as arts) were established. Nikolai Ivanovich Lobachevski (1793 - 1856) published a paper [14] in which he developed a geometry that differed from Euclidean geometry by one axiom only. Namely it used the following negation of the fifth postulate E:

Axiom LB *For some plane π_0 , some line $l_0 \subset \pi_0$ and some point $A_0 \in \pi_0 \setminus l_0$ there exist at least two distinct lines k_1 , k_2 through the point A_0 that do not intersect the line l_0 .*

Great ideas appear in different great minds almost simultaneously: Janos Bolyai (1802 - 1860) had published in 1832 the same ideas in the appendix of his father's book (see [1]). Karl Friedrich Gauss (1777 - 1855) is said to have had investigations in the new geometry but he cannot be praised as much for the discovery not only because he did not publish any result of this kind but also because he had an entirely negative and discouraging attitude towards the discoveries of Janos Bolyai who abandoned mathematics at a young age after being exposed to such an attitude of the "King of mathematicians".

The theory was systematically built up according to strict deductive rules and had no inconsistencies. It was a big surprise at the time and there immediately arose questions as to whether the new geometry was as valuable as the ruling Euclidean geometry, in mathematics, philosophy and the physical world (the space measurements were taken with no instant success, only to be successfully performed after A. Einstein's work in relativity theory).

In the same period differential and projective geometries were developing. The first one led Bernhard Riemann (1826 - 1866) to the introduction (in 1854) of what is nowdays called Riemann spaces. Among them there stood out in particular spaces with constant curvature embracing the parabolic type that corresponds to the Euclidean space, the hyperbolic type corresponding to Lobachevski - Bolyai space and elliptic type corresponding to projective space with suitably chosen metric. Among the first interpretations of Lobachevski - Bolyai Geometry was the one given by Eugenio Beltrami (1835 - 1900). He used a pseudosphere to draw lines l and m, n (asymptotically converging to l) with $m \cap n = P$ and $m \cap l = n \cap l = \emptyset$, and thus interpreting them as "straight lines" obtained the LB axiom. Note that in this case the sum of the angles of a triangle is less than 180° .

A similar interpretation of Riemannian Geometry is to be found in a model of a sphere, where great circles are interpreted as "straight lines" and thus they always intersect (in two points), and the sum of the angles of a triangle is greater than 180° .

On the basis of Beltrami's ideas, Felix Klein (1849–1925) has given in 1871 in [13] basic results on consistency of Lobachevski-Bolyai geometry, whereas David Hilbert formally resolved problems of consistency of both Euclidean and Lobachevski - Bolyai geometry in [9] and [11] respectively.

Hilbert started with a set of *primitive notions* (non-defined intuitive notions such that new notions are built up of these). The primitive notions are: a set $\mathcal{S}$ ("space"), classes of subsets of $\mathcal{S}$ ("lines" and "planes"), ternary relation B and quaternary relation D on $\mathcal{S}$ (B : "betweeness", D : "equidistance"). (At the same time M. Pieri published in [16] and [17] two axiomatic systems of Euclidean geometry that each depended on only one primitive notion.)

Several statements (axioms) give properties of primitive notions that are most likely to be intuitively clear from "everyday experience". The axioms are usually grouped into: *Axioms of incidence* (stating set theoretical relations between points, lines and planes), *axioms of order* (listing properties of the relation B), *axioms of congruence* (about the relation D) and the *axiom of continuity* (enabling the Archimedean property). Geometry determined by these axioms is called *absolute geometry*. If the axiom E or LB is added to the axioms of absolute geometry, then we get respectively *Euclidean* or *Lobachevski-Bolyai geometry*.

It is also assumed that in constructing an axiomatic theory $\mathcal{T}$ use is made of other axiomatic theories (in our case set theory) which are presupposed (i.e. all their primitive notions and axioms are adjoined to those of $\mathcal{T}$).

The main demand on any axiomatic system is its *consistency* i.e. that no antinomy can be derived from the given set of axioms. The question whether some axioms can be shown to be the consequences of the others is that of *independence* of an axiomatic system. Though it may not be as important as consistency, historically it is the investigation of independence of axioms that led to the discovery of new geometries. Proving consistency and independence consists of finding an (outside) consistent model satisfying the axioms (after certain interpretation) of the theory. An axiomatic system is *categorical* if any two of its models are isomorphic i.e. if it has a sufficiently strong axiomatic system determining uniquely its model up to an isomorphism.

The Euclidean geometry has been proved to be categorical and consistent provided the axiomatic system for arithmetic of real numbers is consistent. Namely the three-dimensional Cartesian space $\mathbf{R}^3$ has its own analytic geometry and the notions like points, lines, planes, betweenes and equidistance can be represented as ordered triples of real numbers and certain equations, together with a set and number-theoretical relations between them. Also the Euclidean fifth axiom E can be proved by proving that certain system of equations has a unique solution. The consistency of Lobachevski-Bolyai geometry was proved by interpreting it on Beltrami-Klein space – three-dimensional projective space with its analytic geometry (three-dimensional projective space is a quotient space of $\mathbf{R}^3 \setminus 0$ under the equivalence relation of proportionality of coordinates). The BL axiom holds in this model and it is obvious that Cartesian and Beltrami-Klein models are not isomorphic (they contain contradictory theorems E and LB respectively). Since both of these models are models for absolute geometry we conclude that absolute geometry is not categorical (since it contains at least two non-isomorphic models). On the other hand, as in the case of Euclidean geometry, Lobachevski-Bolyai geometry is also categorical.

We would like to emphasize here that the *consistency of Euclidean and Lobachevski-Bolyai geometry is only relative - dependent on consistency of the arithmetic of real numbers.*

For a detailed treatment of developing foundations of both geometries we recommend [2] to the interested reader.

Axioms for Set Theory

Methods used in a formal mathematical theory $\mathcal{T}$ are characterized by a very precise language, and, since I will content myself with the theory necessary for most of todays mathematics, namely set theory, I will call that language *LST* - the language of set theory. With *LST* we use the rules of logic (the axioms of first order logic, to be precise) and the rules for the formation of complex statements out of elementary ones. (This game must look strange to an outsider, since a non-mathematician friend of mine has recently told me that my mathematics is all squiggles together with lots of equality signs and zeros.)

The rules of the game are called axioms and, in the case of set theory, the most widely used system is the system of Zermelo-Fraenkel axioms (abbreviated as ZF); we give their heuristic list:

1. *Extensionality*: sets having the same elements are equal.
2. *Union*: the union of sets is a set.
3. *Infinity*: there is an infinite set.
4. *Power set*: the collection of all subsets of a given set is likewise a set.
5. *Foundation*: any non-empty set has a member disjoint from that set.
6. *Replacement Scheme*: for any set and a function with that set as domain, its image is also a set.

The replacement scheme as such is infinite and thus the list of axioms is infinite. Moreover it has been proved that no finite collection of LST sentences suffices to axiomatize ZF theory.

Though there may be various systems of axioms suitable for the same or different purposes, apparently not all of them are equally good or good at all. Every axiomatic system however should be tested by the following criteria:

(a) *Consistency*: $\mathcal{T}$ is consistent if there is no statement S such that both S and *nonS* can be derived from $\mathcal{T}$ or equivalently, if there is at least one statement S (formulated in the language of $\mathcal{T}$) that cannot be deducted from $\mathcal{T}$.

(b) *Completeness*: $\mathcal{T}$ is complete if, for every statement S formulated in the language of the theory $\mathcal{T}$, either S or *nonS* can be derived from the axioms of $\mathcal{T}$ according to the deduction rules.

(c) *Independence*: $\mathcal{T}$ has an independent set of axioms if none of its axioms can be derived from the remaining set of axioms.

As it is easy to conclude from the definitions just given that there is a relationship between consistency, completeness and independence, we note the following:

Proposition *A statement S is not provable in $\mathcal{T}$ if and only if $\mathcal{T}$ + nonS is consistent.*

Model Theory

As noted before, the first proofs on consistency and independence of a theory were given at the time of the appearance of non-Euclidean geometries by the use of models. The majority of the results on these metatheorethical questions in set theory were also achieved by the use of models; it is enough to get a (consistent) model for proving that a system of axioms is consistent. A very strong theorems in this area were given by Kurt Gödel (see e.g. [8]).

Gödel Completeness Theorem *If $\mathcal{T}$ is any consistent set of statements then there exists a model for $\mathcal{T}$ whose cardinality does not exceed the cardinality of the number of statements in $\mathcal{T}$ if $\mathcal{T}$ is infinite and is countable if $\mathcal{T}$ is finite.*

There is a Löwenheim-Skolem theorem very similar in content to the just stated theorem . One of the amazing consequences here is that there exists a countable family of sets with the property that if the membership relation is restricted only to those sets, then we get the model for the whole set theory (keep in mind that set theory contains uncountable sets and at the first sight it looks paradoxical that uncountable sets can be pictured in a countable model).

Gödel Incompleteness Theorem *If $\mathcal{T}$ is a consistent, sufficiently strong (i.e. if Peano arithmetic could be built in it), effective list of sentences (i.e. if there is an algorithm for recognizing a sentence from the list), then there is a statement S such that neither S nor nonS can be derived from $\mathcal{T}$.*

Gödel Underivability Theorem *If $\mathcal{T}$ is consistent, sufficiently strong, effective list of sentences, then $\mathcal{T} \nvdash Con(\mathcal{T})$ (i.e. consistency of an axiomatic system cannot be proved from the axioms of that system alone).*

There are numerous applications of the methods used in model theory to the areas outside set theory. At the moment we give an example from non-commutative group theory.

Given a finite number of words $w_i = a_1^{n_1} \cdots a_{k_i}^{n_{k_i}} = 1$, $n_i \in Z$, $a_i \in G$ in a group G, can we determine (find an algorithm) whether another word $w = 1$? Or, equivalently, can w be obtained from w_i by taking multiplication, inverses and conjugation? This problem was translated into arithmetic terms and proved unsolvable in classical axiomatic system of set theory.

We give two more extremely important and useful results from model theory:

Theorem *If a theory $\mathcal{T}$ has an infinite model or arbitrary large finite models, then $\mathcal{T}$ admits models of arbitrarily large cardinalities.*

Compactness Theorem *If every finite subset of an axiomatic system $\mathcal{T}$ has a model, then the whole $\mathcal{T}$ has a model.*

The Axiom Of Choice And The General Continuum Hypothesis

From what has been said so far we understand that one cannot hope to base all conceivable mathematics on a single axiomatic basis and that is the reason that a continuous search for additional axioms is carried out. Various new axioms are being discovered every day. The space allowed makes it possible to list only the two most common ones: The axiom of choice and the general continuum hypothesis.

Axiom of Choice (AC): *For a given collection of sets, there is a set that contains one and only one element of each set of the given collection. This is equivalent to well ordering of any set as well as to the existence of infinite products.*

General Continuum Hypothesis (GCH): *For every infinite set X and every family $\mathcal{F}$ of subsets of X, $\mathcal{F}$ is in one-to-one correspondence either with a subset of X or with a set of all subsets of X. Using the aleph notation it is the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. If $\alpha = 0$, we have the Continuum hypothesis CH.*

Cantor used the axiom of choice as early as 1878 and the continuum hypothesis is also his [3]. Hilbert's first problem (see [10]) was the question of proving AC and CH from the system ZF. Both axiom of choice and the generalized continuum hypothesis however were proved to be independent of ZF. We list a few results from [4,8,5] ($Con(\mathcal{T})$ denotes consistency of $\mathcal{T}$, and $ZFC = ZF + AC$):

(1) $Con(ZF) \Rightarrow Con(ZFC)$,

(2) $Con(ZF) \Rightarrow Con(ZF + nonAC)$,

(3) $Con(ZF) \Rightarrow Con(ZFC + GCH)$,

(4) $Con(ZF) \Rightarrow Con(ZFC + nonCH)$,

(5) $ZF + GCH \Rightarrow AC$,

(6) $ZFC \Rightarrow$ there is a set of real numbers that is not Lebesgue measurable,

(7) $Con(ZF) \Rightarrow Con(ZF + nonAC +$ there is a set of real numbers that is not Lebesgue measurable).

Whereas most mathematicians use AC in their work without questioning it, CH and GCH are not nearly as widely accepted. Moreover there are some very "natural" results following from the negation of GCH or CH.

Algorithmic Unsolvability

At the end of this survey I would like to point out a different kind of independence problems, yet closely related to the ones discussed in the previous section.

Ancient mathematicians have already noted that the ratio of the hypotenuse of an isosceles right triangle to its leg cannot be rational. They have also posed such questions as squaring the circle, doubling the cube or trisecting the angle by the use of only straight edge and compass. All these problems were shown to be impossible to solve, that is to say the axioms of the ruler and the compass do not suffice for making the required constructions (the problems were positively solved by the use of some more powerful devices...). It was also shown that there are polynomials already of the fifth degree whose roots could not be found by means of radicals (this last problem may had been the main step in the discovery of groups).

The most sophisticated among "modern" achievements of this kind is an ingenious solution (by Yuri Matijasevič in [15]) of Hilberth's tenth problem (see [10], also [6,7]).

Hilbert's tenth problem asked for an algorithm testing Diophantine polynomial equations for having integer (or, equivalently, natural) solutions. Although the notion of an algorithm can be precisely defined we will assume an intuitive feeling for the notion and, say that an algorithm is a "procedure" ("circulum vitiosus"!) that could be carried out by a computer in a finitely many steps and a bounded amount of time.

Definitions (a) A set S of ordered n-tuples $(a_1, \dots, a_n)$ of natural numbers is called *Diophantine* if for each such an n-tuple there is a polynomial $P(a_1, \dots, a_n, x_1, \dots, x_m)$, $m > 0$, with integer coefficients such that $P(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has a solution in natural numbers $x_1, \dots, x_m$.

(b) A set S of ordered n-tuples of natural numbers is *listable* (or, in a more latinized version, *recursively enumerable*) if there is a well defined algorithm for making a list of all members of S.

(c) A set $S \subseteq N$ is *computable* if there is an algorithm (of finitely many steps) for deciding whether any natural number belongs to S.

A few examples of Diophantine sets are as follows: integers having an odd divisor, the sets $\{(x, y) : x < y\}$, $\{(x, y) : x \text{ divides } y\}$ .... Some more examples can be obtained through the notion of a Diophantine function. It is such a function that its graph is a Diophantine set; or more precisely: a function f of n variables is a *Diophantine function* if $\{(x_1, \dots, x_m, y) : y = f(x_1, \dots, x_m)\}$ is a Diophantine set. The functions $T(n) = 1 + \dots + n = n(n+1)/2$, $E(n, k) = n^k$, $F(n) = n!$, $B(n, k) = \binom{n}{k}$ are Diophantine.

It is easy to see that every Diophantine set is recursively enumerable. However the following fundamental result (see [15]) shows that the converse is also true:

Theorem *A set is Diophantine if and only if it is recursively enumerable.*

If we express this theorem in, for us more suitable "polynomial form", we have:

Main Theorem *There is a procedure that can be used on any algorithm listing a set S of n-tuples of natural numbers, to get a polynomial P with integer coefficients such that $P(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has a solution in nonnegative integers $x_1, \dots, x_m$ if and only if $(a_1, \dots, a_n) \in S$.*

Now, if a set S is computable it is recursively enumerable but a basic result in recursion theory states that the converse is not true:

Theorem *There is a listable set $S \subseteq N$ which is not computable.*

Corollary *There is a polynomial $P(a, x_1, \dots, x_m)$ such that there is no algorithm for deciding whether $P(a, x_1, \dots, x_m) = 0$ has integral solutions in $x_1, \dots, x_m$, for any given values of the integer parameter a.*

This is a strong negative solution to Hilbert's tenth problem since it states that there is no algorithm for testing solvability of Diophantine equations, even with one parameter only.

Notice that the result does not give the way to find out which Diophantine equations are indeed algorithmically solvable.

Let me now mention a few positive results. One of the consequences of the Main Theorem above is that there exists a polynomial P with integer coefficients containing all prime numbers among its values (there are various examples of such polynomials of less than 12 variables and polynomials of the kind of the fifth degree). For the novelty's sake we list one of such polynomials (see [12]) containing "only" 325 symbols:

Theorem *The set of primes is exactly the positive range (as the variables range over natural numbers) of the following polynomial of the 25th degree and 26 variables :*

$$
\begin{aligned}
&P(\textit{the letters of the English alphabet}) = \\
&\quad (k+2)\{1 - (wz + h + j - q)^2 - [(gk + 2g + k + 1)(h + j) + h - z]^2 \\
&\quad -[16(k+1)^3(k+2)(n+1)^2 + 1 - f^2]^2 - (2n + p + q + z - e)^2 \\
&\quad -[e^3(e+2)(a+1)^2 + 1 - o^2]^2 - [(a^2 - 1)y^2 + 1 - x^2]^2 \\
&\quad -[16r^2y^4(a^2 - 1) + 1 - u^2]^2 - [(a^2 - 1)l^2 + 1 - m^2]^2 - (ai + k + 1 - l - i \\
&\quad -[((a + u^2(u^2 - a))^2 - 1)(n + 4dy)^2 + 1 - (x + cu)^2]^2 \\
&\quad -(n + l + v - y)^2 - [p + l(a - n - 1) + b(2an + 2a - n^2 - 2n - 2) - m]^2 \\
&\quad -[q + y(a - p - 1) + s(2ap + 2a - p^2 - 2p - 2) - x]^2 - \\
&\quad -[z + pl(a - p) + t(2ap - p^2 - 1) - pm]^2\}
\end{aligned}
$$

It is worth mentioning that the methods discovered can be used to reformulate some of the classical problems in mathematics, by getting equivalent statements saying that certain polynomial Diophantine equations have no solutions in nonnegative integers. Among such classical problems are the last

Fermat's theorem, Goldbach's conjecture, four color problem, Riemann conjecture etc. (However the twin-primes conjecture cannot be reduced in this way).

Finally we state here a strong version of Gödel's incompleteness theorem. One can show that the set of theorems in a formalized mathematical theory is listable, whereas the set of unsolvable Diophantine equations is not. Precisely we have:

Incompleteness Theorem *Let T be a theory with its language containing symbols* $0, S, +, *, <$, *with the following properties:*

(a) *T is consistent* ,

(b) *T is listable* ,

(c) *T is strong enough to prove any of the statements of the forms :* $\alpha+\beta=\gamma$, $\alpha * \beta = \gamma$, $\alpha < \beta$, *where* α, β, γ *are among* 0, $S0$, $SS0, \ldots$, *with S being the successor function. Then there is a (polynomial) Diophantine equation* $P(x_1, \ldots, x_m) = 0$, *determined by T, such that* $P = 0$ *does not have a solution in natural numbers, but we cannot prove it within the theory T.*

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MATHEMATICAL EDUCATION

Approaches To School Geometry

Mary Morrin David Murphy Ann O'Shea

Introduction

This article arises from a postgraduate course in geometry given by Professor Barry at U.C.C. As part of the course we undertook some project work on the geometry courses of Georges Papy, Gustave Choquet and Jean Dieudonne. Here we hope to review these three courses and their potential for inclusion in the secondary school curriculum.

First of all, we must ask the question: why teach geometry? One obvious reason for teaching geometry is its application to real life situations and problems. Through the study of geometry children develop practical skills in such areas as measurement, calculations of areas and volumes, use of grids and co-ordinate systems. It also gives them an understanding of the concepts of two-dimensional and three-dimensional space. Clearly geometry has application to topics in mathematics and can indeed be regarded as a unifying theme in the mathematics curriculum. It provides a rich source of visualisation for arithmetical and algebraic concepts. Geometry is essential for mastering calculus and therefore all other fields that have calculus as a prerequisite. A major reason for the inclusion of geometry in the secondary school curriculum is its value as a vehicle for stimulating and exercising general thinking skills, skill in deductive reasoning and problem solving. Through its precise use of language, geometry can also play a part in the development of skills in communication. Therefore, geometry has an important role in the secondary school curriculum.

The next question is: How should we teach geometry in secondary schools? It seems to us that there are two main approaches. One the one hand there is the synthetic approach, which was used by Euclid and later completed and brought to logical perfection by he German mathematician David Hilbert. On the other hand we have an approach to geometry which uses linear algebra. Choquet, Papy and Dieudonne all favour the latter.

We now outline their courses.

Choquet

While Choquet agrees that children benefit from an approach to geometry based on concepts drawn from the real world such as parallelism, perpendicularity and distance he believes that from the mathematician's point of view the most valuable method of defining a plane as a 2-dimensional vector space over $\mathbb{R}$ having an inner product. In order to reconcile these ideas he uses synthetic axioms and sets out to demonstrate the algebraic structure of the plane. Then using the tools of linear algebra he develops the course in geometry.

Choquet's first step is to develop the vector space structure of the plane. His initial axioms are concerned with incidence properties of points and lines and also deal with parallelism. Parallel projection is the natural orders on a line and so he can deal with betweenness of points. His next axiom assumes that parallel projection maps intervals to intervals and therefore, preserves betweeness of points. Choquet makes a strong point that geometry should not be burdened with the task of constructing the real numbers and in his courses he assumes that $\mathbb{R}$ is a totally ordered archimedean commutative field. His next axiom assumes distance on a line.

Now Choquet has both distance and order on every line D, so he can chose any point $o \in D$ and one of the natural orders of D to obtain a pair (D, o) called a pointed line. On this pointed line he can define operations under a unique isomorphism. Thus each line (D, o) is a vector space. Choquet next defines midpoints and postulates that parallel projection preserves midpoints. He can now define a parallelogram as a quadruplet of points (a, b, a', b') such that (a, a') and (b, b') have the same midpoint. Having chosen any point $o \in D$ as origin and writing (Π, o) for the plane Π with origin chosen at o. Choquet defines addition in Π to be the operator $(x, y) \rightarrow x + y$ where $(0, x, x + y, y)$ is a parallelogram. He can show that addition is well defined and proves that $((\Pi, o), +)$ is an abelian group. He next defines scalar multiplication and shows that $(\Pi, 0)$ is a vector space. He uses translations and homothetic maps to show that for any $a, b \in \Pi$ the vector spaces (Π, a) and (Π, b) are isomorphic. Having established the vector (space) structure of the plane Choquet discusses some affine transformations.

The next step is to obtain distance on the plane. Here two further axioms are introduced. The first introduces perpendicularity as an undefined notion and lists its properties. Choquet defines orthogonal projection and the second axiom postulaes that if two segments of equal length but on different lines have the same endpoint then the orthogonal projections of each into the other have the same length. Choquet next chooses an inner product whose symmetry is guaranteed by his last axiom and having established a number of preliminary results, he shows that for all $x, y \in \Pi$, $d(x, y) = \|y - x\|$. At this stage Choquet's course is truly in the domain of Euclidean geometry.

In the remainder of the course Choquet deals with several other topics in geometry. He examines transformations of the plane and pays particular attention to the group of isometries I_o which fix a given point o and to the role of the abelian subgroup R_o (consisting only of rotations) of this group. This lays the groundwork for his definition of angle as rotation and so he obtains immediately that the set of angles with given vertex o is an abelian group. In order to measure angles, Choquet relies on the existence of continuous homorphisms from $\mathbb{R}$ onto the multiplicative group of complex numbers with absolute value one, having shown that the set of angles with given vertex is isomorphic to this group. He treats orientation algebraically and shows how an orientation of the plane can be obtained using either the group of affine transformations or the group of isometries. Choquet also treats elementary trigonometry and the geometric properties of the circle.

Papy

In Papy's opinion, linear algebra provides the best approach to geometry. In his course he uses synthetic axioms to help him represent the plane as a vector space. He begins with three axioms of incidence, then he defines parallelism and direction and his fourth axiom states 'Every direction is a partition of the plane'. At this stage, Papy gives his perpendicularity axiom. He now defines parallel projection as well as the notion of equipollence, which is extremely important in this course.

He proves that equipollence is reflexive and symmetric and by introducing the axiom 'Equipollence is transitive', he deduces that equipollence is an equivalence relation. The equivalence classes are called translations or vectors and the set of translations forms a group under composition. By fixing a point o in the plane Π, every point $x \in \Pi$ will define a vector $\vec{ox}$ and Papy proves that $(\Pi_0, +)$ is also a commutative group.

At this stage, an order axiom is introduced and now half-lines, half-planes etc. can be defined. Papy defines midpoints by using equipollence and so begins the important process of graduation of the line, which will integrate distance into his course. Using transitivity of equipollence and also midpoints Papy can lay off multiples of an interval of the form $p/2^q(p, q \in \mathrm{N})$ along a line. By inserting an archimedean axiom he makes sure he reaches beyond each point on the line and his continuity axiom ensures that every point of the line will be contained in one of his subgraduations. Papy also uses this process of graduation to build up the real numbers and he believes that by introducing the reals in this manner he not only enriches his Geometry but also the concept of a real number.

He defines the abcissa of a point on a line and uses this notion to define aaddition on the reals. Now it is possible to prove that $(\mathbf{R}, +)$ is a commutative group. Papy now defines homothetic maps and uses these to define the multiplication. He proves that $(\mathbf{R}, +, \cdot)$ is a field and that $\mathbf{R}_o$ is a real vector space. Next, Papy defines an inner product and the norm of a vector and so the distance between two vectors can be defined as $\|x - y\|$. We are now dealing with Euclidean Geometry and results such as Pythagoras's Theorem are easily proved using the vector space structure.

From here, Papy goes on to consider the classification of isometries. He discusses the group of angles and the isomorphism between this group and the group of rotations. (He defines angles as 'rotations which have lost their centres'). He also considers the field of similitudes, complex numbers and trigonometry.

Dieudonne

Dieudonne's geometry course is based completely on the concepts of linear algebra— he makes absolutely no concessions to synthetic methods. In fact his main reason for writing this book is to influence secondary school mathematics courses away from synthetic geometry towards a greater acceptance of linear algebra as a method of developing Euclidean plane geometry.

As Dieudonne's will be dealing with vector spaces over the real numbers, he begins by listing a set of axioms for $\mathbf{R}$ which is necessary and sufficient for his course. In particular, these axions make $\mathbf{R}$ into a totally ordered field. Even at this early stage his puritanism intrudes, because instead of a continuity axiom for $\mathbf{R}$, he uses an Intermediate Value property for quadratic and cubic polynomial functions. This lack of a continuity axiom precludes the 'measurement' of angles in the usual sense, wich Dieudonne claims is rightly part of analysis and has nothing to do with algebra or geometry.

He now goes on to define a Eucidean Plane as a two-dimensional vector space over $\mathbf{R}$ with an inner product attached to it. All the standard affine results and properties (including axiom) can be deduced as theorems for the vector space axioms alone, with suitable definitions of line and parallelism. For example, denoting the vector space by E, for $0 \neq b \in E$ he defines a line as

$$L = \{a + \lambda b : \lambda \in \mathbf{R}\} = a + D, \qquad a \in E$$

where $D = \{\lambda b : \lambda \in \mathbf{R}\}$. is called the direction of L. Then $L_1 = a_1 + D_1$ and $L_2 = a_2 + D_2$ are parallel if and only if $D_1 \subset D_2$ (or vice versa). It can be shown that:

(i) For distinct L_1,L_2 lines in E: $L_1 \cap L_2 = \emptyset$ or $L_1 \cap L_2 = \{x\}$.

(ii) Given L_1 a line in E,$c \in E$, then there exists a unique line L_2 in E such that $c \in L_2$ and L_1 is parallel to L_2.

(iii) Through any pair of distinct points $c_1, c_2 \in E$ there is one and only one line.

Using the total ordering on $\mathbf{R}$, he can now define in an obvious way the concepts of midpoint of a segment, betweeness, half-line and line segment.

The standard definitions of translation and affine map are also introduced here viz. if E,F are two dimensional vector spaces, $a \in E$, then $t_a : E \to E$, $t_a(x) = a + x$ is a translation of E by a, while $u : E \to F$ is an affine map if $u = t_b \circ V$, where t_b is a translation of F ($b \in F$ arbitrary) and V a linear map from E to F. We get parallel projections by noting that any two distinct lines intersecting at the origin yield a direct sum decomposition of E (i.e. are supplementary subspaces) and so any $x \in E$ can be decomposed into the sum of two unique elements, one taken from each of the lines.

Placing an inner product on E now makes E into a Euclidean plane. We stress here that any inner product will do and that if two inner products are proportional ($\theta$is proportional to θ' if $\theta = \lambda\theta'$ for some $\lambda > 0$) then they both induce essentially the same Euclidean structure on E. This is not true for non-proportional inner products. Via the inner product we now have immediate access to the Eucidean concepts of orthogonality, perpendicularity, distance in the plane and angle, along with all the standard results from synthetic

Eucliden geometry. For example, with the usual definition of orthogonality, we can deduce a version of Pythagoras' theorem as a one line corollary of Minkowski's inequality, which itself is easy to prove in two dimensions. Two lines are perpendicular if their respective directions are orthogonal subspaces and, using a metric d induced by the inner product, we define a circle, for some fixed $x_0 \in E$ and $\lambda \in \mathbb{R}$, as a set $\{x \in E : d(x_o, x) = \lambda\}$. Dieudonne's treatment of plane geometry finishes with a glance at trigonometry and a development of complex numbers.

Because Dieudonne is intent on introducing linear algebra as well as geometry, some of his constructions are more elaborae than necessary if the main emphasis was on geometry. For example, his introduction of symmetry about the origin develops the concepts of eigenvalue, eigenvector and eigenspace, whereas in a geometric context we could simply define this symmetry map as $u : E \to E, u(x) = -x$. A treatment of plane geometry is given in [1], and this even manages to avoid the explicit introduction of vector space axioms by apprpriate definitions of addition and multiplication in $\mathbb{R}^2$.

This brief outline demonstrates clearly that Dieudonne's approach to geometry differs radically from he synthetic approach and consequently from the methods of treating plane geometry in most elementary school courses.

Conclusions

All three writers are agreed that the ideal way to approaxh geometry is via linear algebra. Consequently, they wish to arrive at a vector space structure as soon as possible. However, here Dieudonne disagrees with the approach adopted by both Choquet and Papy. Dieudonne claims there is no need to 'scaffold' from a synthetic to a vector space structure, and so operates immediately in a vector space.

Choquet and Papy adopt a similar type approach in their courses. They both begin with a set of basic (affine) acioms and graduate develop an algebraic structure on the plane by the addition of more synthetic axioms as required. However, there are areas of difference. For example, whereas Choquet assumes distance on a line and he real numbes, Papy uses graduation of the line to develop these concepts.

They are all agreed, however, that linear algebra gives us what Choquet calls a 'royal road' to geometry.

Before discussing the feasibiity of introducing geometry via linear algebra into second level school courses, it might be fruitful to outline some advantages

of such an approach. One fundamental advantage is that, with linear algebra, 'everything in elementary geometry can be obtained in a very straightforward manner by a few lines of trivial calculation' [3, p.10]. This is a powerful benefit, particularly when coupled with the fact that, in linear algebra, we have a theory 'where everything is ordered naturally around a few simple central ideas which also form the basis for later studies' [3, p.10]; after all, 'there are few mathematical concepts simpler to define than those of vector space and linear mapping' [3, p.11].

Another advantage is that linear algebra 'has become one of the most efficient and central theories of modern mathematics. Its applications now range over a wide and rich field, from the theory of numbers to theoretical physics, analysis, geometry and topology. Consequently there is great advantage to be gained from acquainting the young student at an early stage with the essential principles underlying this branch of mathematics' [3, p.10]. Closely allied to this point is that a linear algebra approach to geometry would bring second level mathematics courses more into line with university teaching [3, p.10].

To show that the advantages of applying linear algebra to geometry are not all one way, we should note that geometric concepts and constructions give 'life' to some of the 'drier' areas of linear algebra and so should make linear algebra more accessible to schoolchildren.

The final two advantages are inextriciably linked: 'From a mathematical point of view, the most elegant, mature and incisive method of defining a plane is as a two-dimensional vector space over the real numbers having an inner product' [2, p.14]. Along with this we have the that the concepts of vector space and inner product, with their developments, give us a logically perfect 'royal road' to geometry which we cannot afford to improve.

We will now look at the question of a linear algebra approach to geometry in schools. If we assume that geometry should be taught in secondary schools (either as part of the core curriculum or as an option extra) it is worth considering if we should remain wih the old synthetic (congruence) approach or whether a change to linear algebra would be beneficial. (Time constraints on the curriculum probably precludes a proper treatment of both.) In the course of this project we gathered some information on the second level geometry syllabuses of about eight countries (West Germany, Sweden, Belgium, France, England, Switzerland, Portugal and Canada) and in most cases (parts of France, in particular, being exceptions) it appeared that synthetic methods are still prefered, with scant and superficial regard given to linear algebra.

The main arguments against a linear algebra approach to plane geometry are outlined in Prof. Barry's article [0]: ' ... to subjugate geometry to linear algebra leads to an impoverishement of geometry. They (i.e. these who favour an old-fashioned (congruence) approach to geometry) value the visual as a helpful rewarding method of reasoning, they are reluctant for pedagogical reasons to impose extra unnecessary layers of abstraction on the young, and they value how mathematics can arise naturally in the small in geometry, growing from simple to more complex situations, in contrast with having to deal from the start with a large, abstract, complex system'. There are essentially two criticisms of linear lagebra here, which can be summarised as follows:

(i) An implicit criticism that the 'visual' is lost when linear algebra is applied to geometry.

(ii) That increased (and unnecessary) abstraction is unhelpful to the young.

We will examine these in order:

(i) The first thing we note is that the visual is not totally lost when we move over to linear algebra. Dieudonne himself recommends the use of instruments such as pantographs and affinographs to instil the idea of the 'geometric transformation of the plane or space as one entity'. He also suggests that the operations of vector addition and scalar multiplication in a two-dimensional vector space can be illustrated 'by a few months working wih squared paper' and this 'should be ample to familiarise pupils with the use of these (vector space) axioms and to prepare acceptance of the fact that the algebraico-geometrical edifice is founded on properties whose practical truth is empirically demonstrable.

However we feel that the whole question of visual aids to reasoning involving children between ages 13 or 14 and 17 or 18 shuold be examined more closely. There is a certain ambiguty in stressing the use of methods which encourage visual aids to reasoning whilst simutaneously telling children that diagrams in no way constitute a proof. The residual effects of this ambiguity are sometimes still apparent even at university. (If we look upon visual aids to reasoning as a subset of intuition, then the case of probability theory is applicable, where, if something is intuitively correct, it is most probably wrong). Clearly at primary level it is essential to use structures which are concrete and easily visualisable, but perhaps at the 1sat, 2nd and 3rd years in secondary schools we should begin to discreetly introduce abstract axiomatic systems, with the emphasis initially on concrete examples. (ii) We have already touched upon this criticism in earlier parts of this paper. Clearly abstract systems cannot be introduced in the early primary years, but children in secondary schools are introduced to mny abstract concepts and seem to be able to deal satisfactorily with them.

The question of the 'necessity' of introducing abstract linear algebra to deal wih geometry in secondary schools is the very point at issue, and to deal with this properly would lead us into a critical discussion of synthetic methods, which would lead us too far afield. Suffice to say that Papy, Choquet and Dieudonne are convinced of the necessity.

A proper discussion of geometry a second level enevitably involves questions of mathematics and pedagogy. Whilst we have some little competence to deal with the former, we are com mpletely ill-equipped to deal with the latter. Our aim therefore is to raise questions and stimulate discussion, the ultimate outcome of which, we hope, will yield a course which will simultaneously satisfy the degree of rigour required by mathematicians as well as being accessible to children in secondary schools.

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NOTES

The Euler φ-Function and Probability

James Ward

The Euler φ-function, $\varphi(n)$, where n is a positive integer, is defined as the number of positive integers less than n which are coprime to n. $\varphi(n)$ may be evaluated using the formula

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

where $p_1, \ldots, p_k$ are the distinct prime divisors of n, so that $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. Readers may be interested in the following derivation of this formula, which was presented by Professor E. Eberlein in his lectures on introductory probability and statistics during the Winter Semester of 1980/81 at the University of Freiburg.

Let us consider the sample space $\Omega = \{1, 2, \ldots, n\}$, and the experiment of selecting at random a number from Ω, all numbers being equally likely to be chosen. Denoting by $|X|$ the cardinality of the set X, we have that for any event X (i.e., any subset of Ω), the probability of X is given by $\Pr(X) = |X|/n$. In particular, if A is the event that a number chosen at random from Ω is coprime to n, then $|A| = \varphi(n)$ by definition. On the other hand we have $|A| = n \Pr(A)$. The formula for $\varphi(n)$ will be established by computing $\Pr(A)$.

Writing $A_i = \{\, r \in \Omega \mid p_i \text{ divides } r \,\}$, then it can be seen that

$$A = A_1^c \cap A_2^c \cap \ldots \cap A_k^c \,.$$

Now $|A_i| = n/p_i$, and so $\Pr(A_i) = 1/p_i$ for $1 \le i \le k$.

We now show that the events $A_1, \ldots, A_k$ are independent. Independence requires that for every subset $i_1, i_2, \ldots, i_r$ of the index set $1, 2, \ldots, k$, we have

$$\Pr(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}) = \Pr(A_{i_1})\Pr(A_{i_2}) \cdots \Pr(A_{i_r}) \,.$$

Now

$$\begin{aligned} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| &= \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_r}} \\ = n\frac{1}{p_{i_1}}\frac{1}{p_{i_2}} \cdots \frac{1}{p_{i_r}} &= n \Pr(A_{i_1})\Pr(A_{i_2}) \cdots \Pr(A_{i_r}) \,, \end{aligned} \tag{1}$$

and also

$$|A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| = n \Pr(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}) \,. \tag{2}$$

Together (1) and (2) establish that the events $A_1, \ldots, A_k$ are independent. It follows that the complementary events $A_1^c, \ldots, A_k^c$ are also independent. Therefore

$$\begin{aligned} \Pr(A_1^c \cap A_2^c \cap \ldots \cap A_k^c) &= \Pr(A_1^c)\Pr(A_2^c) \cdots \Pr(A_k^c) \\ &= \prod_{i=1}^{k} (1 - \Pr(A_i)) = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \,. \end{aligned} \tag{3}$$

Since $A = A_1^c \cap A_2^c \cap \ldots \cap A_k^c$ and $\varphi(n) = n \Pr(A)$ we obtain from (3)

$$\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \,.$$

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Integrating Inverse Functions

Brian M. Dean

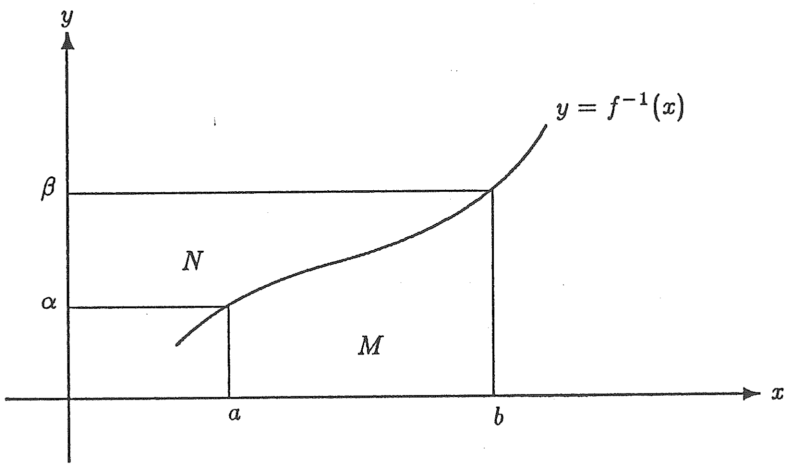
The formula for the derivative of an inverse function is given in every calculus textbook, but is rarely, if ever, pointed out that there is also a formula for the integral of an inverse function. The formula is:

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \tag{1}$$

where $y = f^{-1}(x)$ or for definite integrals

$$\int_a^b f^{-1}(x)\,dx = [xf^{-1}(x)]_a^b - \int_\alpha^\beta f(y)\,dy \tag{2}$$

where $\alpha = f^{-1}(a)$ and $\beta = f^{-1}(b)$. The derivation of this formula is an easy application of integration by parts, taking $u = f^{-1}(x)$ and $v = x$.



The figure gives a graphical interpretation of (2) in the case where f is increasing. The definite integrals give the areas M and N and the term $[xf^{-1}(x)]_a^b$ expresses $M + N$ as the difference of two rectangles. A difference picture is needed for decreasing functions — the details are left to the reader.

With this formula the integrals of many of the standard inverse functions can be computed directly, without working through the details of integration by parts in each case as is usually done. For example,

$$\begin{aligned}\int \arcsin x\,dx &= x \arcsin x - \int \sin y\,dy \\ &= x\arcsin x + \cos y \\ &= = x \arcsin x + \sqrt{1-x^2}.\end{aligned}$$

Thus, if the integral of f is known, we can immediately write down the integral of f^{-1}.

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Conference Reports

Groups in Galway 1987

There were 20 participants at this year's Group Theory Conference, sponsored by the Royal Irish Academy, the Irish Mathematical Society, and University College, Galway, which was held on Friday and Saturday 15th and 16th May 1987, at University College, Galway.

Ted Hurley (U.C.G.) gave the first lecture on "Free Products". He described several free bases for the subgroups $[Y, X, \ldots, X]$ in the free product of a nontrivial group Y and a free group X. Among other applications, it follows that a result proved by P. Hall, about the nilpotency class of a group which stabilises a series of given length in a group, is best possible.

Richard Watson (Maynooth) spoke on "Finite Varieties", which are classes of finite groups closed under the formation of subgroups, quotients, and finite direct products. He discussed the properties of the complete lattice formed by these classes: for instance, he announced that this lattice is modular, and asked whether it is distributive.

Mike Holcombe (Sheffield) lectured on "Algebraic Methods of System Specification". He considered the problem of the design of reliable hardware and software for computers, and defined the concept of an Eilenberg X-machine, which can be regarded as a model of computation.

Roderick Gow (U.C.D.) gave a survey of the problem of realising groups as Galois groups over algebraic number fields. Every finite soluble group can be so realised in infinitely many ways, and at least 19 of the 26 sporadic finite simple groups are known to be realisable over the rational field Q. Hilbert used Riemann surfaces to show that the symmetric and alternating groups are realisable over Q, and Gow has extended work of Schur, who showed that, in certain cases, they are realised as the groups of generalised Laguerre polynomials.

David Simms (T.C.D.) enunciated the axioms for a Poisson algebra, and showed how examples arise naturally in classical mechanics and the theory of partial differential operators, in the symmetric algebra of a finite dimensional Lie algebra, in the theory of symplectic manifolds and the representation theory of Lie groups, and (in a "super" form) from a Clifford algebra.

Edmund Robertson (St. Andrews) delivered the last lecture of the conference on "Nonabelian Tensor Products and Tietze Transformations". He gave a definition, due to Brown and Loday, of the (nonabelian) tensor product of 2 groups equipped with actions on each other, and he stated several pertinent results, which had been suggested by computer calculations.

In addition, Pat Fitzpatrick (U.C.C.) spoke on "Groups with Few Automorphism Orbits".

We would like to thank the lecturers, the sponsors, and the participants for their continued support.

Rex Dark

Cork Operator Conference 1987

This second edition of the Cork Operator Conference, subtitled "Operator theory and operator algebras", happened in Cork on Monday and Tuesday 19–19th May, supported by the Royal Irish Academy, the Irish Mathematical Society and he University College of Cork. From a list of 32 participants, the keynote speakes were Gert Pedersex of Copenhagen, on "Means, extensions and approximations involving unitary operators", Weslaw Zelazke of Warsaw, on "Generation of $B(X)$ by pairs of abelian subalgebras", and John Erdos of London, on "Similarity of nests". 15 other speakers came from America, Canada, Poland, Germany, Japan, Scotland and Ireland.

It is confidently expected that a third edition of the Cork Operator Conference will happen in 1988. [1]

Robin Harte

[1] There is no truth whatsoever in the rumour that it will be subtitled "100 years of Harte and West"!

ILIAM 4 at NIHE, Limerick

The fourth in the series of ILIAM (Information Linkage Between Industry and Applied Mathematics) conferences was held at NIHE, Limerick on 8 May, 1987. The ILIAM meetings are the initiative of the Mathematics Department at NIHE, Dublin.The concept has now developed into a cooperative effort between that Department, the Mathematics Department of the University of Ulsterand the Mathematics Department of NIHE, Limerick.

The objective of ILIAM is to provide over a one day meeting an opportunity for industrialists and academic applied mathematicians to meet and discuss problems of direct interest to industry. With the passing of each conference the range of interaction between academic mathematicians and the industrial community is broadening and deepening. The audiences at the meetings are widely based including engineers and managers from the industrial sector and engineers and mathematicians from the total third level educational sector including the Universities, National Institutes for Higher Education and the Regional Technical Colleges. Some mathematics teachers from the second level system also attend.

The presentations at ILIAM meetings are typically by industrialists describing problems, involving mathematics, whch are of current interest in industry or by mathematicians/engineers from educational institutions describing the application of mathematics to industrially oriented problems.

The programme of ILIAM 4, at NIHE, Limerick on 8 May, 1987 included presentations by Mr. G. Hurley, Electronics Department, NIHE, Limerick with title "Mathematics of Resonanant Phenomena in Transformers"; Mr. M. Quinlan, Manager Director, Microelectronics Application Centre, Limerick with title "MAC – An Interface Between Industry and the Applied Mathematician"; Mr. T. O'Dwyer, Analog Devices Ltd., Limerick with title "Mathematical Modelling of Semiconductor Devices"; Mr. J.J. King, Central Fisheries Board, Dublin with title "Washout of Submerged Vegetation in Irish Lakes"; Professor D. Conniffe, Economic and Social Research Institute, Dublin with title "Experimental Design in the 'Real' and Social Sciences"; Mr. P.J. Shields and Mr. G. Silcock, University of Ulster, Jordanstown with title "Mathematical Models in Fire Safety Engineering"; Dr. J. Carroll, Mathematics Department, NIHE, Dublin with title "Numerical Analysis of Semiconductor Devices"; Mr. C. Humphreys, Howmedica Int., Limerick with title "Computer Integrated Manufacturing".

P.F. Hodnett

Conference Announcements

Members on occasion have expressed a wish for more information on the many Mathematical Conferences that take place throughout the year in many parts of the world. We publish here only a selected few, usually those specially requested by the organisers or of special interest to Irish Mathematicians. The Notices and Bulletins of other Mathematical Societies contain some of these and others. There are to our knowledge two other publications which are devoted entirely to announcements of Conferences and which are very comprehensive and all-embracing. One is the European Mathematical Newsletter looked after by the Mathematisches Forschunginstitut Oberwolfach and the other is the IMU Canberra Circular looked after by Bernhard Neumann. One great thing about these is that they are sent free of charge to those who ask for them! The full addresses are given below.

EUROPEAN MATHEMATICAL NEWSLETTER
Mathematisches Forschunginstitut Oberwolfach,
Geschäftsstelle: Albertstrasse 24,
D-7800 Freiburg im Breisgau.
West Germany.

IMU CANBERRA CIRCULAR

Professor B. H. Neumann,

Division of Mathematics & Statistics,
CSIRO,
Box 1965, GPO Canberra,
ACT 2601, Australia.

OR

Department Of Mathematics,
Institute of Advanced Studies,
Australian National University,
Box 4, GPO Canberra,
ACT 2601, Australia.

Groups In Galway 88

The first Groups in Galway meeting was held in 1978. It has been decided to celebrate the tenth anniversary by adding an extra day to the usual (two-day) format. The 1988 meeting will commence after lunch on Thursday May 26 and conclude after lunch on Saturday May 28.

Further information will be circulated early in 1988. In the meantime, please note the dates and start planning to join in the celebrations!

Any enquiries should be addressed to:

Dr. J. McDermott
Groups in Galway 88
Department of Mathematics
University College Galway
Galway, Ireland.

BAIL V Conference — Shanghai, China.

BAIL V, The Fifth International Conference on Boundary and Interior Layers — Computational and Asymptotic Methods, will be held on June 20–24, 1988 in Shanghai. This conference provides a forum for the discussion of numerical or asymptotic methods for the solution of problems involving boundary or interior layers. The registration fee for participants living outside China is US$260 *if received by January 31 1988* and US$320 thereafter. Hotel costs are US$20 per person per night for persons sharing a twin room and US$30 per person for sole occupancy of a twin room. The cost of meals, including the conference banquet, is US$20 per person per day.

Inquiries from individuals living outside China should be directed to the conference organizer in Dublin:

Pauline McKeever
Conference Management Services
P.O. Box 5, 51 Sandycove Road
Dún Laoghaire, Co. Dublin
Ireland.

Fourth Dublin Conference on Matrix Theory and its Applications

A two-day conference on Matrix Theory and its applications will be held in University College Dublin on March 10 and 11, 1988. Papers are invited on any aspect of linear algebra. The deadline for receipt of abstracts is January 15 1988.

The conference fee is IR£10 (or US$15). All correspondence should be addressed to the conference organizer:

Dr. F.J. Gaines
Department of Mathematics
University College Dublin
Dublin 4, Ireland.

ECMI 88

The first open meeting of the European Consortium for Mathematics in Industry (ECMI) will be held at the University of Strathclyde on August 28–31, 1988.

The scope is wide and includes, for example, the mathematics of semiconductor devices, control theory, nonlinear optimization and modelling, mathematical software etc.

For further details, write to:

Conference Secretariat
Department of Mathematics
University of Strathclyde
Glasgow, Scotland
JANET: CAAS29@UK.AC.STRATH.VAXA

Hyperbolic Problems — Aachen 1988

The Second International Conference on Hyperbolic Problems will be held in Aachen on March 14–18 1988. Significant advances have been made in the last few years in the exact and approximate solution of of systems of nonlinear hyperbolic equations and their applications. The aim of the conference is to bring together scientists in the field for a presentation of recent results and to discuss future research. Further information can be obtained from:

Rolf Jeltsch
Institut für Geometrie und Praktische Mathematik
RWTH Aachen
D–5100 Aachen, Federal Republic of Germany
EARN/BITNET: JELTSCH@DACTH51

International Conference on Radicals — Sapporo, Japan

An international conference on "Radicals — Theory and Applications" is to be held in Sapporo from July 24 to 30, 1988. Further information can be had from:

Prof. Shoji Kyuno
Department of Mathematics
Tohoku Gakuin University
Tagajo, Miyagi 985
Japan

Book Reviews

ATLAS OF FINITE GROUPS by J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson
Clarendon Press, Oxford, 1985, xxxiii+252pp. ISBN 0-9-8531990

The classification theorem for finite simple groups, which was completed around 1980, stated that a finite non-abelian simple group is an alternating group of degree at least 5, a group of Lie type, or one of the 26 sporadic groups.

The first priority of the authors of the Atlas is to print the ordinary character table of as many of these groups as possible since it is their view that this is the most compendious way of conveying information about a group to a skilled reader.

With the infinite families their guideline was "to think how far a reasonable person would go and to go one step further". Thus A_{13} is the largest group considered in the alternating series. A group of given Lie type is specified by two parameters, rank and field size. For low rank a variety of field sizes may be shown while for the highest rank only the smallest field size is shown. For example, the character tables of $PSL(2,q)$ are shown for $q \leq 32$, while for rank 4 only that of $PSL(5,2)$ is shown. All of the sporadic groups are included.

In addition to the ordinary character table the authors present information about the maximal subgroups (nearly always complete), the Schur multiplier, the outer automorphism group, the character table for the corresponding coveing groups and extensions by automorphsims (in most cases), as well as various constructions of the simple group or a near relative.

The book has A3 format pages and is spiral bound. The introduction, pages i to xxxiii, consists of eight chapters in which the simple groups are descibed and explanations are given on how to read the tables and text in the two hundred and fifty two pages of the main body of the Atlas which follow.

The authors seek to reinforce trends in notation that they see as desirable. One of these is Artin's convention that single letters are used for groups that are 'generally' simple, for example, $L_n(q)$ for $PSL(n,q)$, and $S_{2n}(q)$ for $PSp(2n,q)$. This can lead to some confusion, for example, $U_n(q)$ for

$PSU(n,q)$. The chapter on the classical groups is a model of conciseness and comprehensiveness.

I felt the introduction is marred in places by notations being used before their definition. For the expert this poses no problem, but for the neophyte it means that the introduction may have to be read several times.

How accurate is the information in the Atlas? I quote the authors

> " Any complacency we might have had in this regard was rudely shattered when the pre-publication version of the table for the outer automorphism group of the Held group was found to contain an error affecting 22 entries (but obeying the orthogonality conditions)!"

Still, the existence of the Atlas means there can be an agreed starting point for the correction of errors. This together with the existence of the alternative CAS system of Neubuser, Pahlings and Plesken augurs well for the hope of completely correct tables in the near future. The next step would seem to be the production of tables of modular characters, at least for the sporadic groups, and I believe this project is already well under way.

This book is a must for everyone interested in finite groups. Most obviously it collects in one place an enormous amount of information on many of the most important groups. It can accommodate users of various levels of sophistication. it can be used to answer simple questions about, for example, orders of centralizers and numbers of conjugacy classes of elements of the same order or more sophisticated questions about, for example, the possible subgroups a set of elements might generate or about characters of the covering groups. Though only groups of Lie type for low rank and smallish field size are included, these are often surprisingly typical of their families and so can be good pointers in the framing of conjectures about their families.

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THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS
by the late E. L. Ince and I. N. Sneddon FRS
Longman Mathematical Texts, Essex, 1987, 234 pp. St£8.95
ISBN 0-582-44068 8

This book is a welcome edition to the excellent Longman series of Mathematical texts. In it Professor Sneddon has re-presented and extended the material covered by the Late E. L. Ince in his 1939 book published by Oliver and Boyd. The aim of the book is "to provide in a compact form an introduction to the methods of solutions of ordinary differential equations for students of mathematics". This book certainly achieves that aim, although I feel that it provides more than an introduction.

The book begins by considering equations of first order and degree, and in 42 pages presents (a) motivation, (b) a discussion on uniqueness and existence, (c) a discussion on various classifications, and (d) some classic equations. This material of this chapter is presented very concisely and clearly. Examples are used very effectively to illustrate all important principles. One almost feels that if this chapter were expanded that students who understood every detail would already be well on the way to having a good introduction to ordinary differential equations.

Chapter 2 discusses Integral curves. I felt that there could have been more discussion of qualitative solutions, and in particular phase-plane solutions and stability. In Chapter 3 equations of higher degree are discussed, and well presented and Chapter 4 discusses equations of second and higher orders, but one might prefer to see the material of these two chapters presented at a later stage of the book. Chapter 5 discusses linear equations and covers a wide variety of material including Laplace transforms and Green's functions. I thought that the section on Green's functions was very clear and gave a good introduction to a topic that many students find difficult. There is an Appendix on Laplace transforms which covers all that is necessary for this chapter. In Chapter 6, solution in series is considered and Bessel functions and Legendre polynomials are covered in some detail. Chapter 7, while short, covers the basics of Linear systems of equations. The book includes some 375 problems, together with solutions, and interestingly they are not included at the ends of chapters but as a group .

In summary I would recommend that most mathematicians should have this book on their shelves. Any minor faults with the book are due to limitations of space, but I do feel that it was a pity that not even a brief discussion of numerical solution of differential equations was not included, although I am aware that this omission was probably deliberate.

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RIA PROCEEDINGS

Members of the Irish Mathematical Society benefit from a special discount of one–third of the normal price on subscriptions to Section A of the Proceedings of the Royal Irish Academy. Orders may be placed through the IMS Treasurer.

PROBLEM PAGE

Editor: Phil Rippon

First, here is a very pretty problem, which I heard about from Tom Laffey. It appeared in the International Mathematical Olympiad 1986 at Warsaw, and was the hardest problem set, in terms of the total scores of all candidates on individual questions. Nevertheless, several candidates solved the problem and an American student, Joseph Keane was awarded a special gold medal for his solution.

1. *To each vertex of a regular pentagon, an integer is assigned in such a way that the sum of all five integers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$, then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five integers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.*

Next, a problem from John Mason at the Open University, who says that it is known in Maths. Education circles as the Krutetskii Problem. I have also seen it attributed to Lovacz.

2. *A finite number of petrol dumps are arranged around a racetrack. The dumps are not necessarily equally spaced and nor do they necessarily contain equal volumes of petrol. However, the total volume of petrol is sufficient for a car to make one circuit of the track. Show that the car can be placed, with an empty tank, at some dump so that, by picking up petrol as it goes, it can complete one full circuit.*

John Mason also asked the following apparently much harder problem. I am not aware of any reference to this problem in the literature.

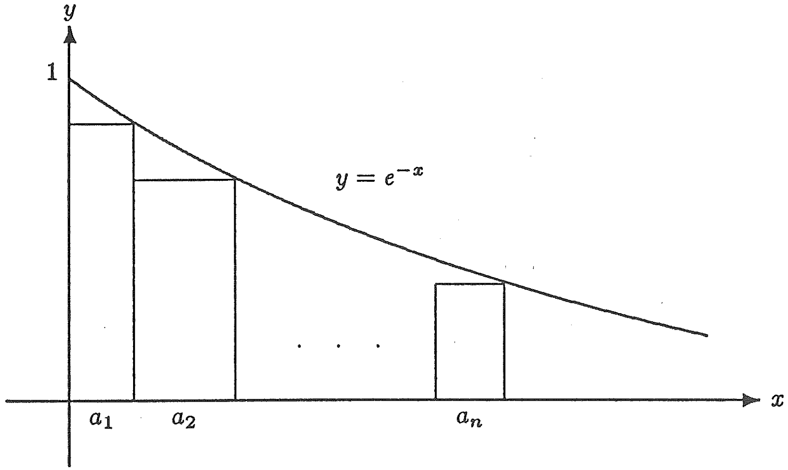
3. *The petrol dumps are arranged as in Problem 2, but this time the total volume of petrol is sufficient for two circuits of the track. Can two cars be placed, with empty tanks, at the same dump so that, by picking up petrol as they go, they can each complete one full circuit in* opposite *directions? (The cars may cooperate in sharing petrol from the dumps.)*

Now, to earlier problems. Finbarr Holland has sent an alternative solution to Tom Carroll's problem:

If $a_n \geq 0$, for $n = 1, 2, \cdots$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{e^{a_1+a_2+\cdots+a_n}} < 1$$

Finbarr's proof can be expressed most succinctly using the following picture:



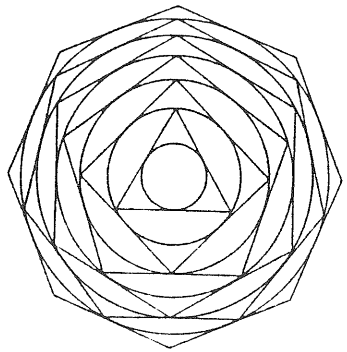
The picture clearly demonstrates that

$$\sum_{n=1}^{\infty} a_n e^{-(a_1+a_2+\cdots+a_n)} < \int_0^{\infty} e^{-x}\, dx = 1.$$

This proof was shown to me recently by Aimo Hinkkanen also.

Next, here are solutions to the problems which appeared in December 1986.

1. The radii of the circles in the following expanding pattern (in which the radius of the innermost circle is 1) tend to a limit which is approximately 8.7.



If at each stage we *double* the number of sides of the escribed polygons, then the limiting radius can be found explicitly. What is it?

In the above diagram the radius of the nth outer circle is

$$[\cos(\pi/3)\cos(\pi/4)\cdots\cos(\pi/(n+2))]^{-1}$$

Since $\cos x \geq 1 - x^2/2$, the infinite product

$$\prod_{n=3}^{\infty} \cos(\pi/n) \geq \prod_{n=3}^{\infty} (1 - \pi^2/2n^2) > 0$$

because

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty\,.$$

If at each stage we *double* the number of sides of the escribed polygons, then the radius of the nth outer circle is

$$\left[\cos(\pi/3)\cos(\pi/6)\cdots\cos(\pi/(3.2^{n-1}))\right]^{-1}.$$

This product can in fact be simplified by a trick due to Euler, based on the half-angle formula:

$$\cos(\theta/2) = \frac{\sin\theta}{2\sin(\theta/2)}, \qquad 0 < \theta < 2\pi .$$

Repeated use of this formula gives

$$\begin{aligned}\cos(\theta/2)\cos(\theta/4)\cdots\cos(\theta/2^n) &= \frac{\sin\theta}{2\sin(\theta/2)}\frac{\sin(\theta/2)}{2\sin(\theta/4)}\cdots\frac{\sin(\theta/2^{n-1})}{2\sin(\theta/2^n)} \\ &= \frac{\sin\theta}{2^n\sin(\theta/2^n)},\end{aligned}$$

Since most terms cancel. If we now use the fact that

$$\lim_{n\to\infty}\frac{\sin(\theta/2^n)}{\theta/2^n} = 1$$

then we obtain

$$\prod_{n=1}^{\infty}\cos(\theta/2^n) = \frac{\sin\theta}{\theta}$$

On substituting $\theta = 2\pi/3$, we get

$$\prod_{n-1}^{\infty}\cos(\pi/(3.2^{n-1}) = \frac{\sin(2\pi/3)}{2\pi/3} = \frac{3\sqrt{3}}{4\pi}.$$

Hence the limiting outerradius in this problem is $4\pi/3\sqrt{3} \simeq 2.42$.

2. If $f(x) = p(x)e^x$, where p is a quadratic with integer coefficients, is it possible for f, f' and f'' to have rational zeros?

This problem was told me by John Reade at Manchester. Unfortunately, for setters (and solvers!) of curve-sketching problems, the answer is 'no'. Here is John's solution to the problem.

We may assume that

$$p(x) = x^2 + \lambda x + \mu, \qquad \lambda,\ \mu \text{ rational}.$$

The conditional for p to have rational zeros is

$$\lambda^2 - 4\mu = s^2, \qquad s \text{ rational}.$$

Now

$$f'(x) = (p(x) + p'(x))\,e^x = \left(x^2 + (\lambda+2)x + \mu + \lambda\right)e^x,$$

and so the condition for f' to have rational zeros is

$$(\lambda+2)^2 - 4(\mu+\lambda) = t^2, \qquad t \text{ rational},$$

which reduces to

$$s^2 + 4 = t^2. \tag{1}$$

Similarly, the condition for f'' to have rational zeros reduces to

$$t^2 + 4 = u^2, \qquad u \text{ rational}. \tag{2}$$

Multiplying (1) and (2) by the common denominator of s^2, t^2, u^2, we obtain

$$\begin{cases} a^2 + d^2 = b^2 \\ b^2 + d^2 = c^2 \end{cases} \qquad a,\ b,\ c,\ d \text{ integers} \tag{3}$$

Now we use the method of infinite descent to show that thre are *no solutions* to (3). The idea is to show that if a solution a, b, c, d to (3) exists, then it is possible to construct a smaller solution α, β, γ, δ to (3). Repeated application of this argument leads to a contradiction, and so (3) has no solutions.

We need a well-known result on Pythagorean triples (see almost any book on number theory).

Lemma *If $x^2 + y^2 = z^2$, where x, y, z are comprime, then z is odd and exactly one of x, y is even. If x is even, then*

$$x = 2uv, \quad y = u^2 - v^2, \quad z + u^2 + v^2,$$

for some comprime integers u, v.

If the equations (3) have a solution a, b, c, d, then we may assume that a, b, c, d have no common factors, and hence that each pair is comprime. The lemma now implies that b, c are odd, so that d is even and a is odd. Hence

$$h = \frac{1}{2}(a+c), \qquad k = \frac{1}{2}(c-a)$$

are positive integers, and it is easy to check that

$$h^2 + k^2 = b^2, \qquad 2hk = d^2 .$$

Also h, k are comprime with each other and with b.

Now suppose that h is even (the argument is similar if k is even). Then, by the lemma,

$$h = 2uv\,, \quad k = u^2 - v^2\,, \quad b = u^2 + v^2\,,$$

where u, v are comprime. Thus

$$d^2 = 2hk = 4uv(u-v)(u+v)\,,$$

where u, v, $u-v$, $u+v$ are pairwise comprime. Hence each of u, v, $u-v$, $u+v$ is a perfect square, say

$$u = \beta^2\,, \quad v = \delta^2\,, \quad u - v = \alpha^2\,, u + v = \gamma^2 .$$

Since this gives

$$\begin{aligned} \alpha^2 + \delta^2 &= \beta^2 \\ \beta^2 + \delta^2 &= \gamma^2 \end{aligned}$$

and $\beta^4 = u^2 < b$, the proof that (3) has no solutions is complete.

3. Which integers can be expressed as the sum of two or more consecutive positive integers?

A quick check of the integers 1, 2, 3, $\cdots$, 10, say, suggests the conjecture that all positive integers except the sequence 1, 2, 4, 8, $\cdots$ can be expressed in this way: $3 = 1+2$, $5 = 2+3$, $6 = 1+2+3$, $7 = 3+4$, $9 = 4+5$, $10 = 1+2+3+4$.

This conjecture turns out to be true and, surprisingly, there is an 'easy' proof.

First we check that any sum of consecutive integers must have an odd factor (> 1). Here is one way to verify this:

The sum of an odd number $2m+1$ of consecutive integers can be written in the form:

$$\underbrace{(n-m)+\cdots+\underbrace{(n-1)+n+(n+1)}_{\vdots}+\cdots+(n+m)} = (2m+1)n\,; \qquad (4)$$

the sum of an even number $2n$ of consecutive integers can be written in the form:

$$\underbrace{(m-n+1)+\cdots+\underbrace{m\;+(m+1)}_{\vdots}+\cdots+(m+n)} = (2m+1)n\,. \qquad (5)$$

This proves that the integers $1, 2, 4, 8, \cdots$ cannot be expressed as sum of two or more consecutive integers.

At first sight, it seems much harder to prove that all other positive integers *can* be expressed in this way. However, any other positive integer can be written in the form $(2m+1)n$, where m, n are positive integers. Moreover, we have either $n-m \geq 1$ or $m-n+1 \geq 1$. Hence $(2m+1)n$ can be expressed as the sum of consecutive positive integers using either (4) or (5).

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Additions to:

"CONSOLIDATED INDEX OF ARTICLES"

The last issue contained a consolidated index of all the articles published in the IMS Bulletin (formerly the IMS Newsletter) since its inception. However,the first issue was unavailable at the time this index was compiled.

Issue 1 (1978) of the IMS Newsletter contained the following articles:

- T.J. Laffey, "*Polynomial Identities and Central Identities for Matrices*", pages 11–16.
- F.J. Gaines, "*Matrices in Europe in the 13th and 14th Centuries*", pages 17–23.

The following article was also omitted from the Consolidated Index:

- T. Hurley, "*Matrix: Computer Assisted Mathematics Teaching*", IMS Bull. 16, 1986, 61–67.
