PROBLEM PAGE

Recent issues of the London Mathematical Society Newsletter have carried items by David Singmaster on the following pattern, which was first contemplated, apparently, by Kepler.

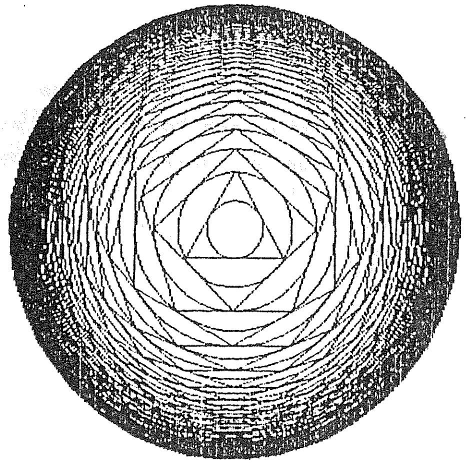

FIGURE 1

It is not hard to convince yourself that the radii of the expanding circles do *not* tend to infinity, but the exact value of the limiting radius seems to be unknown. If the inner radius is 1 then the limiting radius is approximately 8.7 and this number has been calculated to 55 decimal places by Herman P. Robinson in *Popular Computing* (Oct. 1980).

The first problem this time is related, but much easier.

1. If, at each stage, we *double* the number of sides of the escribed polygons, then the limiting radius can be found explicitly. What is it?

I heard the next problem from John Reade at Manchester. He came across it while setting questions on curve-sketching.

2. If $f(x) = p(x)e^x$, where p is a quadratic with integer coefficients, is it possible for f, f' and f'' to have rational zeros?

The final problem must be an old chestnut, but it has a very pretty answer.

3. Which integers can be expressed as the sum of two or more consecutive positive integers?

Now for the solutions to two earlier problems.

1. A rectangle R is partitioned into a finite number of rectangles $R_1, R_2, \ldots, R_n$, each of which has the property that at least one side is of integer length. Prove that R has the same property.

According to Bob Vaughan at Imperial College, this problem originated in France. It was mentioned by J-M. Deshouiller at the conference in honour of Professor K. Roth at Imperial College in 1985, and has spread far and wide since then.

It has a remarkable solution based on the following fact: if:

$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\},$$

then

$$\iint_R e^{2\pi i(x+y)}\,dxdy = 0,$$

if and only if at least one of the numbers $b-a$ and $d-c$ is an integer. This is true because

$$\iint_R e^{2\pi i(x+y)}\,dxdy = \left[\int_a^b e^{2\pi ix}dx\right]\left[\int_c^d e^{2\pi iy}dy\right]$$

$$= \left[\frac{e^{2\pi ib} - e^{2\pi ia}}{2\pi i}\right]\left[\frac{e^{2\pi id} - e^{2\pi ic}}{2\pi i}\right].$$

To solve the rectangle problem, we simply write

$$\iint_R e^{2\pi i(x+y)}dxdy = \sum_{k=1}^{n}\iint_{R_k} e^{2\pi i(x+y)}dxdy = 0,$$

since each of $R_1, R_2, \ldots, R_n$ has at least one side of integer length. Hence R has the same property.

Notice that the method generalises to higher dimensional boxes.

2. A rod moves so that its end points lie on a convex curve Γ_1 in $\mathbb{R}^2$ and a point P, which divides the rod into lengths a and b, then describes a closed curve Γ_2. Prove that the area lying between Γ_1 and Γ_2 is πab.

This strange fact is known as Holditch's Theorem. It can be proved using Green's formula. First parameterize as follows.

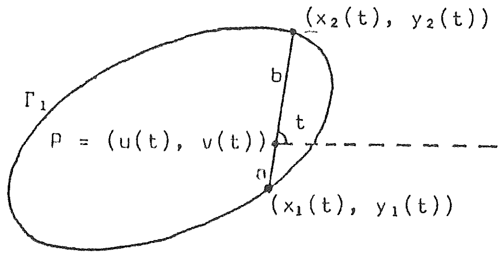

FIGURE 2

If we assume, as we may, that $a+b = 1$ then $x_2 - x_1 = \cos t$, $y_2 - y_1 = \sin t$, and

$$u = bx_1 + ax_2, \qquad v = by_1 + ay_2.$$

The areas A_1, A_2 lying inside Γ_1, Γ_2 can then be obtained from Green's formula as follows:

$$A_1 = \tfrac{1}{2}\int_0^{2\pi}(x_1\dot{y}_1 - y_1\dot{x}_1)dt = \tfrac{1}{2}\int_0^{2\pi}(x_2\dot{y}_2 - y_2\dot{x}_2)dt;$$

$$A_2 = \tfrac{1}{2}\int_0^{2\pi}(u\dot{v} - v\dot{u})dt$$

$$= \tfrac{1}{2}\int_0^{2\pi}[(bx_1 + ax_2)(b\dot{y}_1 + a\dot{y}_2) - (by_1 + ay_2)(b\dot{x}_1 + a\dot{x}_2)]dt$$

$$= (a^2 + b^2)A_1 + \tfrac{1}{2}ab\int_0^{2\pi}(x_1\dot{y}_2 + x_2\dot{y}_1 - y_1\dot{x}_2 - y_2\dot{x}_1)dt$$

$$= A - \tfrac{1}{2}ab\int_0^{2\pi}[(x_2 - x_1)(\dot{y}_2 - \dot{y}_1) - (y_2 - y_1)(\dot{x}_2 - \dot{x}_1)]dt$$

$$= A_1 - \tfrac{1}{2}ab\int_0^{2\pi}(\cos^2 t + \sin^2 t)dt$$

$$= A_1 - \pi ab.$$

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