PROBLEM PAGE

First of all, here's a simple arithmetical problem.

1. The recurring decimal $0.\dot{0}0\dot{1}$ represents a rational number. How long is the recurring block of digits in the square of this number?

Next, a problem sent in by Des MacHale.

2. Prove that at least one of the numbers $\pi + e$, πe, is transcendental.

Finally, a quickie on infinite series.

3. Suppose that $a_n \geq 0$, for $n = 1, 2, \ldots$. How large can

$$\sum_{n=1}^{\infty} \frac{a_n}{e^{a_1+a_2+\ldots+a_n}}$$

be?

Now here are the solutions to some earlier problems.

1. If $1 < p \leq 2$ and $\alpha = \frac{\pi}{2p}$ show that

$$\left(\frac{\cos\theta}{\cos\alpha}\right)^p \geq 1 + \tan\alpha\cos p\theta, \qquad 0 \leq \theta \leq \alpha.$$

Since there is equality when $\theta = \alpha$, it is enough to prove that

$$-\sin\theta(\cos\theta)^{p-1} \leq -\sin\alpha(\cos\alpha)^{p-1}\sin p\theta, \qquad 0 \leq \theta \leq \alpha$$

which follows immediately if the function

$$\frac{\sin\theta(\cos\theta)^{p-1}}{\sin p\theta}$$

is decreasing for $0 < \theta \leq \alpha$. On differentiation this reduces

to proving that, for $0 < \theta \leq \alpha$,

$$1 \leq (p-1)\tan^2\theta + p\frac{\tan\theta}{\tan p\theta}$$

$$\Leftrightarrow \sec^2\theta \leq p[\tan^2\theta + \frac{\tan\theta}{\tan p\theta}]$$

$$\Leftrightarrow \sin p\theta \leq \frac{p}{2}[(1 - \cos 2\theta)\sin p\theta + \sin 2\theta\cos p\theta]$$

$$\Leftrightarrow (2-p)\sin p\theta \leq p\sin(2-p)\theta.$$

Since $0 \leq 2-p < p$ and $p\theta \leq \pi/2$, this final inequality holds because $\sin t/t$ is decreasing for $0 < t \leq \pi/2$. The proof is complete.

Remarks

1. The special case $\theta = 0$ can be written as:

$$1 \geq (\cos\alpha)^p + (\cos\alpha)^{p-1}\sin\alpha, \qquad 1 < p \leq 2,$$

where $\alpha = \pi/2p$. Is there a simpler proof of this?

2. The inequality appears in a paper by Matts Essen ("A Superharmonic Proof of the M. Riesz Conjugate Function Theorem, *Ark. for Mat.*, 22 (1984) 241-249). It is needed to show that a certain function is superharmonic and thus to prove the M. Riesz theorem (with the best possible constant). The 'elementary' proof of the inequality, given above, was first found by Wolfgang Fuchs.

2. Suppose that $0 = \phi_1 \leq \ldots \leq \phi_n \leq \pi$, that $A = [\sin(|\phi_i - \phi_j|)]$ and that $||A|| = \max\{||Ax|| : ||x|| = 1\}$. Show that

$$||A|| \leq \cot\left(\frac{\pi}{2n}\right)$$

and characterize the case of equality.

This problem was kindly sent in by Bob Grone of Auburn University. He also supplied the following solution, which depends on two results which are not particularly well known. I'll state these results first and discuss them in more detail after showing how they solve the problem.

Result 1 An nxn matrix A with non-negative entries has an eigenvalue λ with the following properties:

(i) $\lambda \geq |\mu|$, for all eigenvalues μ of A, and

(ii) λ has an associated eigenvector $x = [x_1, \ldots x_n]^T$ of unit norm for which

$$x_i \geq 0, \qquad i = 1, \ldots, n.$$

Result 2 If $a_1, \ldots, a_n$ are the lengths of the sides of a plane closed polygon and θ_{ij} is the angle between the positive directions of the sides a_i and a_j, then the area of the polygon is

$$\tfrac{1}{2} \sum_{i \neq j} a_i a_j \sin\theta_{ij}.$$

To solve the problem, note that $A = [\sin|\phi_i - \phi_j|]$ is symmetric and so one can form an orthogonal basis for $\mathbb{R}^n$ of eigenvectors of A. Using this one easily sees that

$$||A|| = \lambda = x^T A x = 2 \sum_{i>j} x_i x_j \sin(\phi_i - \phi_j),$$

where λ is the eigenvalue and x the eigenvector described in Result 1.

According to Result 2, the sum on the right is exactly twice the area of the centro-symmetric 2n-gon whose first n consecutive edges have lengths $x_1, \ldots, x_n$ and are inclined at angles $\phi_1, \ldots, \phi_n$ to the x-axis. By the isoperimetric inequality, this is at most twice the area of the regular 2n-gon with perimeter $2(x_1 + \ldots + x_n)$. Hence

$$||A|| \leq \frac{(x_1 + \ldots + x_n)^2}{n} \cot(\tfrac{\pi}{2n}).$$

Among all positive unit vectors x, the maximum value of $(x_1 + \ldots + x_n)^2$ equals n, and this is obtained uniquely at $(1/\sqrt{n}, \ldots, 1/\sqrt{n})$. Thus

$$||A|| \leq \cot(\tfrac{\pi}{2n}),$$

with equality if and only if $\phi_{i+1} = i\frac{\pi}{n}$, $i = 1, \ldots, n$.

Result 1 belongs to the Perron-Frobenius theory of non-negative matrices (see, for example, E. Seneta, "Non-Negative Matrices and Markov Chains, Springer). To prove it one puts

$$K = \{x \in \mathbb{R}^n : ||x|| = 1,\ x_i \geq 0,\ i = 1, \ldots, n\}$$

and

$$r(x) = \min \frac{1}{x_i} \sum_j a_{ij} x_j \qquad x \in K.$$

One then shows that r is bounded above on K and that

$$\lambda = \sup\{r(x) : x \in K\}$$

is attained at an eigenvector x with associated eigenvalue λ.

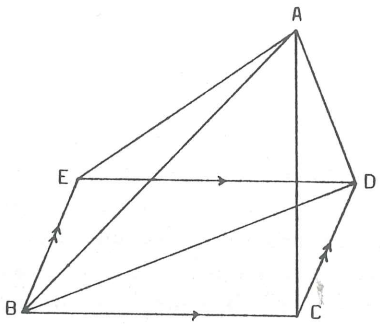
To complete the proof note if $y = [y_1, \ldots, y_n]^T$ is any eigenvector of unit norm with associated eigenvalue μ then

$$|\mu y_i| \leq \sum_j a_{ij} |y_j|, \qquad i = 1, \ldots, n.$$

and so

$$|\mu| \leq \min \frac{1}{|y_i|} \sum_j a_{ij} |y_j| \leq \lambda.$$

Result 2 is easily proved by induction using the following fact about areas:



Area ΔABC = Area ΔBCD + Area ΔAED.

Jim Clunie points out that Result 2 appears in Hobson's 'Plane Trigonometry' and also remarks that 'They don't write books like that anymore!"

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