CONVEXITY AND SUBHARMONIC FUNCTIONS

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This article gives a simple account of some of the ways in which notions of convexity are related to the study of subharmonic functions. Several recent results in this area are included in the discussion. The article is based on a lecture given at the December 1985 meeting of the DIAS Mathematical Symposium.

1. Notation

We shall be concerned with Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, points of which are denoted by $X = (x_1, \ldots, x_n)$. We write $|X| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$, and denote the open ball of radius r centred at X by $B(X,r)$. The closure and boundary of a subset E of $\mathbb{R}^n$ will be denoted respectively by $\overline{E}$ and ∂E.

2. Harmonic Functions

A function u on an open subset ω of $\mathbb{R}^n$ is called harmonic if it is twice continuously differentiable and satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \equiv 0.$$

(Harmonic functions arise naturally in gravitation, electrostatics, hydrodynamics and the theory of analytic functions). Alternatively, letting $M(u,X,r)$ denote the mean value of u over the sphere $\partial B(X,r)$ whenever $\overline{B}(X,r) \subset \omega$, a function u is harmonic in ω if and only if:

(i) $-\infty < u < +\infty$ in ω;

(ii) u is continuous in ω;

(iii) $\overline{B}(X,r) \subset \omega \Rightarrow u(X) = M(u,X,r)$.

3. Subharmonic Functions

By subdividing (i) - (iii) above we arrive at the dual notions of sub- and superharmonicity.

SUBHARMONIC FUNCTION

(ia) $-\infty \leq u < +\infty$ in ω [$u \not\equiv -\infty$ on any component of ω];

(iia) u is upper semicontinuous (u.s.c.), i.e. $\{X \in \omega : u(X) < a\}$ is open $\forall a \in \mathbb{R}$;

(iiia) $\overline{B}(X,r) \subset \omega \Rightarrow u(X) \leq M(u,X,r)$.

SUPERHARMONIC FUNCTION

(ib) $-\infty < u \leq +\infty$ in ω [$u \not\equiv +\infty$ on any component of ω];

(iib) u is lower semicontinuous, i.e. $\{X \in \omega : u(X) > a\}$ is open $\forall a \in \mathbb{R}$;

(iiib) $\overline{B}(X,r) \subset \omega \Rightarrow u(X) \geq M(u,X,r)$

Such functions arise naturally in many situations. For example, if f is analytic in $\mathbb{C}$, then $\log|f|$ is subharmonic. Again, the gravitational potential energy due to a mass distribution is superharmonic. We can immediately make the following observations:

(I) u is subharmonic if and only if $-u$ is superharmonic;

(II) u is harmonic if and only if both u and $-u$ are subharmonic;

(III) if u,v, are subharmonic and $a,b > 0$, then $au + bv$ is subharmonic.

An equivalent formulation of the definition of a subharmonic function is obtained if we replace (iiia) above by:

(iiia') for any open set W with compact closure in ω, and for any continuous function h on $\overline{W}$ which is harmonic in W and satisfies $h \geq u$ on ∂W, we have $h \geq u$ in W.

It is this condition which accounts for the name subharmonic.

4. One-Dimensional Potential Theory

Laplace's equation for the real line is simply $d^2u/dx^2 \equiv 0$, so that harmonic functions are just linear functions of the form $ax + b$ $(a,b \in \mathbb{R})$. In view of (iiia') above, the concept of a subharmonic function on a subset of $\mathbb{R}$ is equivalent to the idea of a convex function. Thus subharmonic functions are a generalization to higher dimensions of convex functions. This explains why notions of convexity recur so frequently in the study of subharmonic functions.

5. Spherical Means

Spherical means of functions have played a fundamental role in potential theory since the pioneering work of F. Riesz [6] in 1926. It is natural to consider how $M(u,X,r)$ behaves as a function of r. Riesz showed that, if $n = 2$, then $M(u,X,r)$ is convex as a function of $\log r$ and, if $n \geq 3$, then $M(u,X,r)$ is convex as a function of r^{2-n}. The functions $\log|X|$ $(n = 2)$ and $|X|^{2-n}$ $(n \geq 3)$ arise as solutions of Laplace's equation in $\mathbb{R}^n \setminus \{0\}$.

Thus, when we modify a subharmonic function (by taking its mean over a sphere of radius r and fixed centre) so that it depends only on one variable (r), convex functions reappear. It is worth pointing out that the same convexity properties hold for

$$\sup\{u(Y) : |Y - X| = r\}, \quad \log M(e^u,X,r), \text{ and}$$

$$\{M(u^p,X,r)\}^{1/p} \text{ for } u \geq 0 \text{ and } p > 1.$$

6. Composition Properties

If we begin with functions of one real variable, we can make the following simple observations of functions:

$$[\text{Convex}] \circ [\text{Linear}] = [\text{Convex}]$$

$$[\text{Increasing Convex}] \circ [\text{Convex}] = [\text{Convex}].$$

('Increasing' is to be interpreted in the wide sense, i.e. non-decreasing). It is well known that these properties carry across to higher dimensions as follows:

$$[\text{Convex}] \circ [\text{Harmonic}] = [\text{Subharmonic}] \qquad (1)$$

$$[\text{Increasing}] \circ [\text{Subharmonic}] = [\text{Subharmonic}]. \quad (2)$$

However, it has only recently (see [3], [5]) been noticed that this is a special case of the more general, but equally elementary, result stated below (for applications, see [3]).

THEOREM 1. The function $v\phi(u/v)$ is subharmonic in each of the following cases:

(i) u is harmonic, v is positive and harmonic, ϕ is convex;

(ii) u is subharmonic, v is positive and harmonic, ϕ is convex and increasing;

(iii) u is subharmonic, v is positive and superharmonic, ϕ is convex, increasing, and $\phi(x) = 0$ for $x \leq 0$.

By taking $v = 1$, it is clear that (i) and (ii) include (1) and (2) above. The proof is quite short and we outline it below.

LEMMA 1. If $\{u_\alpha : \alpha \in I\}$ is a family of subharmonic functions on ω and $\sup_\alpha u_\alpha$ is u.s.c. and less than $+\infty$, then $\sup_\alpha u_\alpha$ is subharmonic in ω.

Proof of Lemma: $\bar{B}(X,r) \subset \omega \Rightarrow u_\beta \le M(u_\beta,X,r) \le M(\sup_\alpha u_\alpha,X,r)$

$$\Rightarrow \sup_\alpha u_\alpha \le M(\sup_\alpha u_\alpha,X,r),$$

so $\sup_\alpha u_\alpha$ satisfies conditions (ia) - (iiia) of Section 3.

Sketch Proof of Theorem: Corresponding to each part of the theorem, ϕ can be written as:

(i) $\phi(x) = \sup\{ax + b : a,b \in \mathbb{R} \text{ s.t. } at + b \le \phi(t) \quad \forall t \in \mathbb{R}\}$;

(ii) $\phi(x) = \sup\{ax + b : a \ge 0,\ b \in \mathbb{R} \text{ s.t. } at + b \le \phi(t)$ $\forall t \in \mathbb{R}\}$;

(iii) $\phi(x) = \sup\{ax + b : a \ge 0,\ b \le 0 \text{ s.t. } at + b \le \phi(t)$ $\forall t \in \mathbb{R}\}$.

Thus $v\phi(u/v)$ can be written as

$$\sup_{a,b} v[a(u/v) + b] = \sup_{a,b} [au + bv]$$

and $au + bv$ is subharmonic for the appropriate values a,b in each of the three cases. It is quite easy to check that $v\phi(u/v)$ is u.s.c., and clearly $v\phi(u/v) < +\infty$, so the result now follows from Lemma 1.

Remark: Theorem 1 and its proof transfer easily to the axiomatic setting of harmonic spaces, and so can be applied to sub-solutions of a wide class of elliptic and parabolic p.d.e.'s. This is particularly interesting because (1) and (2) do not hold for harmonic spaces, the reason being that the constant function 1 is not necessarily harmonic in the general setting.

7. Convex Domains

Let $\Omega \ne \mathbb{R}^n$ be a domain (connected, non-empty open set) in $\mathbb{R}^n$, and let u be the signed distance function given by

$$u(X) = \begin{cases} -\text{dist}(X,\Omega) & \text{if } X \in \bar{\Omega} \\ \text{dist}(X,\Omega) & \text{if } X \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

The following recent result is due to Armitage and Kuran [11].

THEOREM 2. The function u is subharmonic in $\mathbb{R}^n$ if and only if the domain Ω is convex.

The "if" part of the result is straightforward and was already known, at least implicitly. For example, when $n = 2$, let L denote an arbitrary straight line $a_L x_1 + b_L x_2 = c_L$ in $\mathbb{R}^2 \setminus \Omega$, $(a_L^2 + b_L^2 = 1)$, and let u_L be the signed distance function from L given by $u_L = \pm(a_L x_1 + b_L x_2 - c_L)$, the sign being chosen so that $u_L < 0$ in Ω. Since each u_L is harmonic, $u = \sup_L u_L$ and u is real-valued and continuous, it follows from Lemma 1 that u is subharmonic in $\mathbb{R}^n$.

The "only if" part requires a longer argument and is genuinely new. A surprising fact about this result is that more can be said when $n = 2$:

THEOREM 3. The function u is subharmonic in $\Omega \subset \mathbb{R}^2$ if and only if Ω is convex.

Armitage and Kuran give a counterexample to show that Theorem 3 fails in higher dimensions. For example, when $n = 3$, let Ω be the torus obtained by rotating the disc $D = \{(0,x_2,x_3) : (x_2 - 2)^2 + x_3^2 < 1\}$ about the x_3-axis. Then it can be shown that u is subharmonic in Ω yet Ω is clearly not convex.

8. Generalized Means

Convexity properties of spherical means of subharmonic functions (Section 5) have analogues for "weighted means" of such functions over other surfaces. To take a simple example, if u is subharmonic in the upper half-plane $\{(x_1,x_2) : x_2 > 0\}$ and $u \le 0$ on the x_1-axis, then (using polar coordinates)

$$r^{-1}\int_0^\pi \sin\theta\, u(r,\theta)d\theta$$

is convex as a function of r^{-2}.

More generally, various authors over the past 15 years have shown convexity properties for weighted means over the boundaries of half-balls and truncated cones (of varying radii) and bounded cylinders (of varying height or varying radii). In fact, these separate studies have recently ([4]) been unified into a general convexity theorem. The general mean is defined in terms of harmonic measure, and the surface over which it is defined is obtained as the level surface of the quotient of two harmonic functions. For example, in the above case of the half-plane, the appropriate harmonic functions are x_2 and $x_2 r^{-2}$, so the semi-circular means arise as integrals over level surfaces of r^{-2} and convexity is in terms of r^{-2}.

Finally, we remark that convexity properties are not confined to integrals of subharmonic functions over bounded surfaces, for (see [2], for example) if $u \geq 0$ is subharmonic on $\mathbb{R}^{n-1} \times (a,b)$ and does not grow "too quickly" as $|X|$ becomes large, then

$$x_n \mapsto \int_{\mathbb{R}^{n-1}} u(X)\,dx_1 dx_2 \ldots dx_{n-1} \qquad (a < x_n < b)$$

is a convex function provided it is finite on a dense subset of (a,b).

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