CAPACITIES, ANALYTIC AND OTHER

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(1.1) Let E be a compact subset of $\mathbb{C}$. If f is analytic on $S^2 - E$, then it has the Laurent expansion

$$f = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

near ∞, where S^2 is the Riemann sphere. The (Ahlfors) *analytic capacity* of E is the non-negative number

$$\gamma(E) = \sup |a_1(f)|$$

where f runs over all functions, analytic on $S^2 - E$, and bounded by 1 in modulus. A compact set E has $\gamma(E) = 0$ if and only if E is *removable* for all bounded analytic functions, i.e. if and only if given U open and $f: U \sim E \to \mathbb{C}$, analytic and bounded, there exists an analytic continuation of f to U. For open sets U, $\gamma(U)$ is defined as

$$\sup \{\gamma(E) : E \subset U,\ E \text{ compact}\}.$$

For arbitrary sets $A \subset \mathbb{C}$, the *outer analytic capacity* $\gamma^*(A)$ is defined as

$$\inf \{\gamma(U) : A \subset U,\ U \text{ open}\}.$$

(Readers interested in more details should consult [H] for references.)

(1.2) Analytic capacity plays a key role in the theory of uniform rational approximation (or, what amounts to the same thing, holomorphic approximation) in one variable. Let $\mathcal{O}(E)$ denote the set of functions, holomorphic near E. For X compact in $\mathbb{C}$, let $R(X)$ denote the set of uniform limits on X of elements of $\mathcal{O}(X)$. Vitushkin showed that a necessary and sufficient condition that all functions continuous on X belong

to $R(X)$ is that

$$\gamma(U \sim X) = \gamma(U)$$

for all open sets U (or, equivalently, for all open discs U). The capacity γ, in combination with another, the *continuous analytic capacity* α, provides a similar resolution (also due to Vitushkin) of the problem of which X have

$$R(X) = \{f : f \text{ is continuous on } X \text{ and analytic on int } X\}.$$

See [G] for other uses of γ in connection with $R(X)$.

(1.3) There are two important open questions about γ. The first is to give a reasonable "real-variable" characterisation of the γ-null sets. For instance, Vitushkin has conjectured that $\gamma(E) = 0$ if and only if almost all projections of E on lines have outer length zero. Thanks to some work of Havinson, Calderon and others, we know this is true for σ-rectifiable sets, and for those totally unrectifiable sets known to be γ-null [M]. This problem is particularly irritating because the bounded analytic functions are practically the only "reasonable" class of analytic functions for which the null sets lack a real-variable description. For instance, see [C]. The only significant exception are the Smirnov E_p classes, but they do not count, because, when defined, they have the same null sets as γ [H].

The second problem is whether γ is *quasi-subadditive*, i.e. whether there exists a universal constant $\kappa > 0$ such that

$$\gamma(E_1 \cup E_2) \le \kappa\{\gamma(E_1) + \gamma(E_2)\}$$

whenever E_1 and E_2 are compact in $\mathbb{C}$. There is a sizeable logjam in uniform holomorphic approximation theory because of this problem. For example, if E is compact, with $\gamma(E) = 0$, and $f: S^2 \to \mathbb{C}$ is continuous, do there exist functions $f_n: S^2 \to \mathbb{C}$, tending uniformly to f on S^2, holomorphic wherever f is and on a neighbourhood of E? If γ is quasi-subadditive, the answer is yes. If γ were subadditive, one could define a special topology (the "analytic-fine topology") on $\mathbb{C}$, finer than the Euclidean topology, that ought to be especially helpful for studying $R(X)$. This topology might provide the real answer to E. Borel's dream of the perfect notion of analytic function.

(1.4) The most penetrating work on the subadditivity problem is in [D]. Davie showed that quasi-subadditivity would follow from the statement:

$$\gamma^*(E \cup F) \leq \gamma(E) + \kappa(E)\gamma(F)$$

wherever E is compact and F is open, where $\kappa(E) > 0$ is independent of F. We know that

$$\gamma(E \cup F) \leq \gamma(E) + \kappa(E)\gamma(F)$$

wherever E and F are compact. It may not seem like much of a gap, but there it is.

In what follows, we shall present another formula for $\gamma(E)$, and use it to cast a little light on the subadditivity problem. It will become clear that subadditivity is just another version of the only "real" problem in analysis, which is how to handle

$$\int_{-1}^{1} \frac{f(t)}{t}\, dt.$$

(2.1) Dolzenko generalised the concept of analytic capacity. Suppose B is a Banach space of functions on $\mathbb{C}$, such that $\mathcal{D} \hookrightarrow B$, $\mathcal{D} \hookrightarrow B^*$, and the inclusions are continuous. Here $\mathcal{D} = \mathcal{D}(\mathbb{C},\mathbb{C})$ denotes the space of test functions. We assume that, if B has a predual B_*, then $\mathcal{D} \hookrightarrow B_*$, continuously. Also, we assume $f \in B \Rightarrow \bar{f} \in B$. The *analytic B-capacity* of a compact $E \subset \mathbb{C}$ is

$$\gamma_B(E) = \sup |a_1(f)|$$

where f runs over all functions in the unit ball of B that are analytic on $S^2 - E$.

Examples are $B = L_p$ (with respect to area measure $\mathcal{L}^2$), C (for continuous and bounded), Lipα, lipα, BMO, VMO, C^k (= bounded continuous derivatives up to order k), some weighted L_p spaces, Sobolev spaces, etc.

(2.2) The number $a_1(f)$ equals

$$\frac{1}{2\pi i}\int_\Gamma f(\zeta)\,d\zeta$$

whenever Γ is a rectifiable contour around E, in the usual sense. A more entertaining formula is

$$a_1(f) = -\frac{1}{\pi}\int f(z)\,\frac{\partial\psi}{\partial\bar{z}}\,d\mathcal{L}^2(z) = \frac{1}{\pi}\left\langle \psi, \frac{\partial f}{\partial \bar{z}} \right\rangle$$

where $\psi \in \mathcal{D}$ is any test function with $\psi = 1$ on a neighbourhood of E. This follows from Green's formula. It suggests the natural way to generalise γ_B from the Cauchy-Riemann operator to other differential operators.

Let E be a compact subset of $\mathbb{R}^d$, let B be a Banach space of functions on $\mathbb{C}$, and let $\mathcal{E}(\mathbb{R}^d, \mathbb{C})$ be the Schwartz space of C^∞ functions from $\mathbb{R}^d$ to $\mathbb{C}$. Let $L : \mathcal{E}(\mathbb{R}^d, \mathbb{C}) \to \mathcal{E}(\mathbb{R}^d, \mathbb{C})$ be a linear differential operator with C^∞ coefficients. Choose $\psi \in \mathcal{D}(\mathbb{R}^d, \mathbb{C})$ with $\psi = 1$ on a neighbourhood of E, and define

$$\gamma_B^L(E) = \sup \left| \int f(x) L^*\psi(x)\, d\mathcal{L}^d(x) \right|,$$

where f runs over all elements of the unit ball of B which satisfy $Lf = 0$ on $\mathbb{R}^d - E$, in the (weak) sense of distributions. The value of the integral does not depend on the choice of ψ, for such f. This concept embraces those capacities used by

Hedberg, Polking, Bagby, and others [HE] in connection with various approximation problems. The classical Newtonian capacity is $\gamma^{\Delta}_{L_\infty}$, where Δ is the Laplacian.

(2.3) The technique of the *dual extremal problem* is based on the following fact, which may be proved by using the Hahn-Banach theorem.

Duality Lemma. *Let* B_0 *be a subspace of a Banach space* B, *and let* $A \in B^*$.

(1) Then $\sup \{|Af| : f \in B_0, \ ||f||_B \leq 1\} = \operatorname{dist}(A, B_0^\perp)$.

(2) *If* B *has a predual* B_*, *if* B_0 *is* $B_1^\perp$, for some subspace $B_1 \subset B_*$, *and if* $A \in B_*$, *then*

$$\sup \{|Af| : f \in B_0, \ ||f||_B \leq 1\} = \operatorname{dist}(A, B_1).$$

This lemma allows us to turn to an extremal problem in one Banach space into a corresponding problem in the dual, or in the pre-dual (if there is a pre-dual). This technique has been put to good use in the past, but still has plenty of energy left. Our present purpose is to apply it to get formulae for the kind of capacities described above, so as to cast some light on the subadditivity problem.

(2.4) Applying part (1) of the Duality Lemma gives the formula

$$\gamma^L_B(E) = \inf_S ||L^*\psi - S||_{B_*}$$

where S runs over all elements of B^* such that

$$\left.\begin{array}{l} f \in B \\ Lf = 0 \text{ off } E \end{array}\right\} \Rightarrow Sf = 0.$$

If B has a predual B_* (and $D \hookrightarrow B_*$ is continuous), part (2) gives the nicer formula

$$\gamma^L_B(E) = \inf_\phi ||L^*\psi - L^*\phi||_{B_*}$$

where ϕ runs over all test functions supported on $\mathbb{R}^d \sim E$. Recalling that ψ is any given test function with $\psi = 1$ near E, we conclude that

$$\gamma^L_B(E) = \inf \{||L^*\phi||_{B_*} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E\}.$$

(2.5) Applying this formula to classical analytic capacity, we get

$$\gamma(E) = \frac{1}{\pi} \inf \{||\tfrac{\partial\phi}{\partial\bar z}||_{L_1} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E\}.$$

(2.6) Applying it to the analytic capacity associated to $B = L_p$ (the "analytic p-capacity" of Sinanjan), we get

$$\gamma_{L_p}(E) = \frac{1}{\pi} \inf \{||\tfrac{\partial\phi}{\partial\bar z}||_{L_q} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E\}$$

for $1 < p < \infty$, where q is the conjugate index to p. This B has the property that B is mapped continuously to itself by the Beurling transform:

$$(Tf)(z) = \frac{1}{\pi} \int \frac{f(\zeta)}{(\zeta - z)^2} \, d\mathcal{L}^2(\zeta),$$

where the integral is interpreted as a limit in B norm of principal value integrals of smooth approximation to f. The theory of the continuity properties of this and similar integral operators is known as the Calderon-Zygmund theory [A,S]. The operator T has the property that

$$T \frac{\partial\phi}{\partial\bar z} = \frac{\partial\phi}{\partial z}$$

for all $\phi \in \mathcal{D}$, so that if T maps $B \to B$ continuously, we deduce that γ_B is comparable to the real-variable capacity

$$\inf \{||\tfrac{\partial\phi}{\partial x}||_{B_*} + ||\tfrac{\partial\phi}{\partial y}||_{B_*} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E\}.$$

Apart from L_p $(1 < p < \infty)$, the spaces Lipα and BMO are Beurling-invariant dual spaces, so this argument also applies to their analytic capacities. In all three cases, this real-variable formula gives a proof of quasi-subadditivity. For instance, for BMO we get

$$\gamma_{BMO}(E) \sim \inf \{ \| |\nabla\phi| \|_{L_1} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E \},$$

where ~ means "is within constant multiplicative bounds of". It makes no difference to restrict to real-valued ϕ, and we get

$$\gamma_{BMO}(E) \sim \inf \{ \|\phi\|_{W^{1,1}} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E \}$$

$$= \inf \{ \|h\|_{W^{1,1}} : h \in W^{1,1}, \ h = 1 \text{ near } E \}$$

$$= \inf \{ \|h\|_{W^{1,1}} : h \in W^{1,1}, \ h \geq 1 \text{ near } E \};$$

which is obviously subadditive. Here $W^{1,1}$ denotes the Sobolev space of L_1 functions with L_1 distributional derivatives. See [V].

(2.7) This method extends to other hypoelliptic operators. Suppose L^* has an inverse $P : \mathcal{D} \to \mathcal{E}$ such that $PL\phi = \phi$ whenever $\phi \in \mathcal{D}$. For instance, the Cauchy transform does this for $\frac{\partial}{\partial \bar{z}}$, and, more generally, convolution with a fundamental solution does it for elliptic constant-coefficient L.

Suppose L has order m. Denoting the partial derivative associated to the multi-index j by D_j, we may ask about the continuity properties with respect to B of the operator D_jP, for $|j| \leq m$. If all these map B_* continuously into B_*, then $\gamma_B^L(E)$ is comparable to the real-variable capacity

$$\inf \{ \sum_{|j| \leq m} \|D_j\phi\|_{B_*} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E \}.$$

This works for constant-coefficient elliptic operators, with $B = L_p$ $(1 < p < \infty)$, Lipα, BMO, Lip$(k+\alpha)$, some Sobolev spaces, etc. The associated γ_B^L are then subadditive.

(2.9) If L has *real-valued* coefficients, then γ_b^L is a real-variable capacity even if B is not Beurling invariant. For instance,

$$\gamma_{L_\infty}^{\Delta}(E) = \inf \{ \|\Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E \}$$

$$\inf \{ \|\Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \ \phi = 1 \text{ near } E, \ \phi \text{ real} \}$$

$$\inf \{ \|\Delta\phi\|_{L_1} : \phi \in \mathcal{D}, \ \phi \geq 1 \text{ near } E, \ \phi \text{ real} \}.$$

This is pretty clearly subadditive.

(2.10) The upshot is that among the usual crop of elliptic operators L and dual spaces B, the case $L = \bar{\partial}$ and $B = L_\infty$ is practically the only one we cannot handle with ease. And we cannot handle it at all.

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RAZMYSLOV AND SOLVABILITY

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The exponential growth in the number of active mathematicians in the present era is sometimes illustrated by the remark that there are as many mathematicians alive today as have lived - and died - since classical times. A less picturesque but more interesting indicator of mathematical activity is the rapidity with which well known conjectures and problems, sometimes of long standing, are being resolved. A recent article in the *Newsletter* (No. 11) by David Lewis on the Merkuryev-Suslin Theorem illustrates this point, and the present article (also expository, also concerned with Russian work) provides another example.

INTRODUCTION

Many readers will be familiar with, or at least aware of, the Burnside Problem in group theory, namely: must a group be finite if it is finitely generated and has exponent k? Having exponent k means that the group elements all satisfy the law $x^k = 1$ and some element has period precisely k. The problem was stated in 1902 [1], and answered negatively in 1968; an outline of developments and a bibliography, may be found in [4] and [3]. The story is by no means complete, and many problems remain open concerning these groups, but one problem concerning solvability has been settled completely by the work of Ju. P. Razmyslov in Moscow.

Let B_k denote the Burnside Variety of all groups satisfying the law $x^k = 1$; let $B_{k,n}$ represent the free group of rank n in B_k (then the n-generator groups of exponent k are just the quotient-groups of $B_{k,n}$). It has been known for many years (> 25) that: