PROBLEMS

Jim Stack from Waterford R.T.C. sent a solution to the parking problem (Issue No. 8) which arrived just too late for Issue No. 9. His solution was similar to the one given last time and he mentions that the idea is to be found in an article by D.E. Knuth ("Computer Science and Its Relation to Mathematics", *Amer. Math. Monthly*, April 1974) which discusses the retrieval of information from memory locations, using the analogy of musical chairs instead of car-parking.

Now for solutions to the most recent problems.

1. *The Plank Problem.* Does there exist a positive integer n such that a closed disc of diameter 1 can be covered by fewer than n planks of width $\frac{1}{n}$?

To see that the answer is 'no', recall that when two parallel planes meet a sphere of diameter 1 the area between the planes on the surface of the sphere is πd, d being the distance between the planes.

If the original disc is taken to lie on the equator of a sphere then the vertical projection of any plank of width $\frac{1}{n}$ meets the sphere's surface in a region with area at most $\frac{\pi}{n}$. If a collection of planks covers the disc then their vertical projections cover the surface of the sphere and so at least n planks are required.

Remark. The "generalised plank problem" deals with convex sets with diameter 1 (i.e. the smallest door they can be pushed through has width 1), and the planks need not have equal widths. It is to be shown that the sum of the plank widths is at least 1 and this was done by Bang, but I'm afraid I've mislaid the reference.

2. *The Planet Problem.* A finite number of equal spherical planets are in outer space. A region on the surface of one of the planets is called hidden if it is invisible from any of the other planets. Find the total area of the hidden regions.

In fact their total area is equal to the area of the surface of a single planet. This is much easier to visualise than to write down - but here goes.

Suppose that the planets have radius 1 and let $\underline{p}_1, \ldots, \underline{p}_n$ denote their centres (I shall identify position vectors and points throughout). Let $S_0 = \{\underline{u} : |\underline{u}| = 1\}$ so that

$$S_i = \{\underline{p}_i + \underline{u} : |\underline{u}| = 1\}, \quad i = 1, \ldots, n,$$

denote the surfaces of the planets. Also let $S = \bigcup_{i=1}^{n} S_i$, and for each $\underline{u} \in S_0$ put

$$[\underline{u}] = \{\underline{p}_i + \underline{u} : i = 1, \ldots, n\}.$$

The proof consists of noticing the following facts.

(i) The set $E = \{\underline{u} : \underline{p}_i + \underline{u}$ is visible from $\underline{p}_j + \underline{u}$, some $i \neq j\}$ lies in a finite union of circles on S_0, and so has area zero.

(ii) A point $p \in S$ is hidden $\Leftrightarrow$ there is a plane π through $\underline{p}$ such that $S \setminus \{\underline{p}\}$ lies entirely to one side of π.

(iii) For any $\underline{u} \in S_0$ there is a plane π orthogonal to $\underline{u}$ and a non-empty set $F \subseteq [\underline{u}]$ such that $S \setminus F$ lies entirely to one side of π. If $|F| > 1$ then $\underline{u} \in E$ and if $|F| = 1$ then $F = \{\underline{p}\}$ where $\underline{p}$ is hidden.

(iv) Each set $[\underline{u}]$ contains at most one hidden point. Thus, for each $\underline{u} \in S_0 \setminus E$ the set $[\underline{u}]$ consists of exactly one hidden point and the proof is complete.

Remark. It looks as if the result remains true when the spheres have different radii, if we replace "areas" by "solid angles".

3. *Group Theory Problem* from September 1983 Issue (page 74). Let n be a natural number. A group G is said to be n-Abelian if $(ab)^n = a^n b^n$ for all $a,b \in G$. Find all the values of n for which

(i) G is n-Abelian implies that G is $(n+1)$-Abelian.

(ii) G is n-Abelian implies that G is $(n-1)$-Abelian.

Solution (i) G is n-Abelian implies that G is $(n+1)$-Abelian if and only if $n = 2$ or 3.

Proof Clearly G is 2-Abelian if and only if G is Abelian. All groups are 1-Abelian, so G is 1-Abelian does not imply that G is 2-Abelian. Also trivially if G is 2-Abelian, G is 3-Abelian.

Now assume that G is 3-Abelian. Then $(ab)^3 = a^3b^3$, so by cancellation $(ba)^2 = a^2b^2$ for all $a,b \in G$. Then

$$(ab) = [(ab)^2]^2 = (b^2a^2)^2 = a^4b^4, \text{ so } G \text{ is 4-Abelian.}$$

Next assume that $n > 3$. We claim that for each such n there is a group which is n-Abelian but not $(n+1)$-Abelian. Let G be a non-Abelian group of exponent $n-1$. Then $(ab)^{n-1} = a^{n-1}b^{n-1}$ and $(ab)^n = ab = a^n b^n$, for all a,b G. Thus G is $(n-1)$-Abelian and n-Abelian. If G is $(n+1)$-Abelian, then G is k-Abelian for three consecutive values of k, so by a well-known result, G is Abelian, a contradiction.

(ii) The above example shows that G n-Abelian implies G is $(n-1)$-Abelian only in the trivial cases $n = 2$ and $n = 1$.

Just one new problem this time from me:

Problem. For $1 \leq p \leq 2$ show that

$$(1+x^2)^p \leq 1 + (2^p - 2)x^p + x^{2p}, \quad x \geq 0.$$

What happens for other values of p ?

This inequality (which I learnt from Jim Clunie) is related to a problem in ℓ^p spaces posed by Finbarr Holland. It is a very special case of a conjectured inequality in n variables, which I'll say more about next time.

Finally, a short test on Linear Analysis and Ring Theory from Robin Harte.

Problem 1 Suppose $T:X \to Z$ and $S:Y \to Z$ are bounded linear mappings between normed linear spaces, and write row(T,S) for the mapping

$$(x,y) \to Tx+Sy : X \times Y \to Z.$$

(a) If T and S are bounded below and row(T,S) is one-one, does it follow that row(T,S) is bounded below?

(b) If $\alpha > 0$ and $\beta > 0$ are such that, for each $x,y \subset X,Y$,

$$||x|| \leq \alpha||Tx|| \text{ and } ||y|| \leq \beta||Sy||,$$

and if row(T,S) is one-one, does it follow that

$$\max(||x||, ||y||) \leq ||\alpha Tx + \beta Sy|| \ ?$$

Problem 2 Suppose $T:X \to Y$ and $S:Y \to Z$ are bounded linear mappings between normed linear spaces:

(a) if ST is one-one, T is bounded below and S is relatively open, does it follow that ST is bounded below?

(b) if $\alpha > 0$ and $\beta > 0$ are such that, for each $x,z \in X,Z$,

$$||x|| \leq \alpha||Tx|| \text{ and } z \in S(Y) \Longrightarrow z \in \{Sy : ||y|| \leq \beta||z||,$$

and if ST is one-one, does it follow that ST is bounded below?

Problem 3 Suppose A is a ring, with identity 1, and a_1, a_2, b_1, $b_2 \in A$ satisfy

$$b_1a_1 = 1 = a_2b_2 :$$

does it follow that $a_2a_1ca_2a_1 = a_2a_1$, with possibly $c = b_1b_2$?

Problem 4 Suppose A and B are rings with identity 1, and suppose that $T:A \to B$ is additive and satisfies

$$T(1) = 1 \text{ and } T(A^{-1}) \subseteq B^{-1},$$

where A^{-1} and B^{-1} are the groups of invertible elements in A and B. Does it follow that T is multiplicative? Does it at least follow that T has the Jordan property

$$T(a_2a_1+a_1a_2) = T(a_2)T(a_1) + T(a_1)T(a_2) \ ?$$

Problem 5 Suppose A is a ring with identity, and write $A_{n\times n}$ for the matrices over A. If $a \in A_{n\times n}$ has mutually commuting entries $a_{ij} \in A$ and a left inverse $b \in A_{n\times n}$, must its determinant $|a|$ have a left inverse in A?

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A SHORT DICTIONARY OF MATHEMATICAL TERMS

Des MacHale

As a young graduate student I was frequently perplexed by certain words and phrases which cropped up again and again in the research papers which I attempted to read. Conventional Mathematics dictionaries gave me no help whatsoever, but experience has since taught me the true meaning of many of these expressions. For the benefit of those who find themselves in the same position, I offer a selection, in the hope that it will stimulate others to contribute to this sadly neglected area of mathematical education.

1. *The proof is left as an exercise*: I've lost the envelope on which I jotted this down, but it seemed reasonable at the time.

2. *While the results of Holland are relatively deep*: Holland once mentioned a paper of mine in his references.

3. *Formal Process*: I can't understand this for the life of me, but it seems to work.

4. *By far the most significant results in this field are due to Hurley*: Hurley is likely to referee this paper.

5. *I wish to thank the referee for a number of useful suggestions*: The old meanie cut me down from twenty pages to a miserable four.

6. *While only partial results have been obtained*: I've made no progress at all with this problem but I figured I could get at least one publication from it.

7. *It is well known that*: I'm not quite sure how to prove this and I'm hanged if I'm going to the trouble of finding out who first discovered it.