GROUPS AND TREES

John McDermott

For some years now, the M.Sc. Algebra course in U.C.G. has included a unit on group presentations based mainly on the first half of Johnson's book [4]. One of the attractions of the book is its emphasis on computational aspects of the subject such as coset enumeration. On the occasions that I taught this unit, I have experimented with the use of graph-theoretical ideas in the presentation of supplementary material and also to provide an alternative approach to some of the topics in the text. The graph theory involved uses little more than basic concepts (in particular, a course on graphs is not a pre-requisite); but the intuitive "geometric" framework it provides is, I believe, helpful to the student. I made some comments along these lines in a talk at the DIAS Symposium in December 1982. In this note, which is based on that talk, I outline some of the graph-theoretical approaches to group presentation.

A *graph* is usually defined to be a (non empty) set V of *vertices* (points), some pairs of which are joined by *edges*. In the context of group presentations we should, strictly, talk about *directed multigraphs*: that is, an edge may be directed from one vertex to the other, and there may be several edges between two given vertices. Moreover, the graph may be *coloured*: its vertices and/or its edges may have colours (labels) attached; these are usually elements of G, some group related to the graph. In particular, we adopt the convention that an edge from x to y labelled g is implicitly an edge from y to x coloured by $g^{-1} \in G$.

The classical examples of graphs related to group presentations are *Cayley diagrams* and *Schreier diagrams*. Let G be a group generated by a,b, ... ; the corresponding Cayley diagram is the graph having the elements of G as vertices and for each generator, such as a, an edge from x to y coloured with a if $y = xa$. If H is a subgroup of G, the corresponding Schreier diagram has the cosets of H in G as its vertices and, for each generator a, an edge labelled a from Hx to Hy if $Hxa = Hy$. There are examples in Fig. 1.

(i) (ii)

FIGURE 1

In (i) we have the Cayley diagram for $G = \langle a \mid a^3 = 1 \rangle$, the cyclic group of order 3; for convenience, we use i to denote the vertex a^{i-1}, and an arrow on each edge indicating its direction.

In (ii) we have the Schreier diagram of G with respect to H, where $G = \langle a,b \mid a^3 = b^2 = (ab)^2 = 1 \rangle$, the non-abelian group Sym (3) of order 6, and $H = \langle b \rangle$, a subgroup of order 2. The solid edges are each coloured a in the direction of the arrow, the dotted edges are coloured b (in either direction) and the vertex Ha^{i-1} is denoted by i.

A Schreier diagram is particularly useful in the study of a group presentation. We can read off from it not only the index of H in G - which is just the number of vertices in the diagram - but also the images of the generators of G in the permutation representation of G on the cosets of H. For example, we can see in (ii) that

$$a \longmapsto (123) \quad \text{and} \quad b \longmapsto (23)$$

One may view the construction of a Schreier diagram as a pictorial implementation of the technique of *coset enumeration*; and with a little elaboration it may also be used to describe the process of finding a presentation for H in terms of the given presentation for G. There is a nice informal description of some of these ideas in the first section of Chapter VIII of the book [1] by Bollobas.

Now we need a couple of definitions. A *path* from x to y in a given graph is what intuition suggests, essentially a sequence $x_0, x_1, \ldots, x_k$ of vertices such that $x_0 = x$, $x_k = y$ and each pair x_{i-1}, x_i is joined by an edge. If some of the edges are directed, we do not insist that the direction is from x_{i-1} to x_i; moreover, if there are several edges between x_{i-1} and x_i it is necessary to indicate which edge is intended in the path. A *tree* is a graph in which, given any distinct vertices x and y, there is a unique path from x to y; equivalently, a tree is *connected* (there is a path between any two vertices) *and contains no cycle* (path with $x_k = x_0$, $k > 0$, not using any edge twice). It can be shown, by successive deletion of edges from cycles, that any connected graph contains a *spanning tree*, that is a tree using all the original vertices. For example, the graph in Fig. 1(ii) is obviously connected; it contains cycles such as 1231, and two of its spanning trees are shown in Fig. 2.

(i) (ii)

FIGURE 2

In the last fifteen years, there has been a new and striking use of graphs to illustrate the theory of group presentations, in particular the basic constructions such as free groups and free products. This is the Bass-Serre theory of *groups acting on trees*, and the basic reference is Serre's book [6]. The starting point is the following easy observat-

ion. The Cayley diagram T(F) of a free group F with respect to a set of free generators is a tree: the "generating" property says that T(F) is connected, and the "free" property says that it has no cycles. Moreover, F acts by left multiplication as a group of automorphisms of T(F) and the action is free (no vertex or edge is fixed by any non-identity element of F). This property actually characterises free groups:

THEOREM: A group F is free if and only if it acts freely on some tree.

Remark: The proof of the "if" part is non-trivial, but it is possible to extract a reasonably simple account from Sections 2 and 3 of the first chapter in [6]. The proof actually produces a set of free generators for F. A key technical point is that given a spanning tree T_0 of the natural quotient graph T/G, there is a subtree of T which projects isomorphically onto T_0.

An immediate pay-off is *Schreier's subgroup theorem*:

COROLLARY: If F is a subgroup of a free group F_0 then F is free.

Proof: The theorem ensures that F_0 acts freely on some tree T. Since F is a subgroup of F_0, it acts freely on the same T, and hence it is free.

Remark: In the situation of the corollary, consider F as acting of $T(F_0)$, and choose a spanning tree T_0 in the Schreier diagram of F_0 with respect to F. Then the free generators for F produced by the theorem are in one to one correspondence with those edges of the diagram that do not belong to T_0. Using this fact, it is easy to establish the *Schreier index formula* for the rank (number of free generators) of F when the index $|F_0 : F|$ is finite. Moreover, there is a link here with the classical approach to these matters. The spanning tree T_0 corresponds to a certain *Schreier transversal* for F_0 in F: choose as coset representative for a given vertex i the product of the colours on the unique path in T_0 from 1 to i. For example, recall the groups G and H used to illustrate the idea of a Schreier diagram in Fig. 1(ii). If we interpret G as F_0/N, where F_0 is the free group on a and b and N is the normal subgroup of F_0 generated by a^3, b^2 and $(ab)^2$, we may consider Fig. 1(ii) as the Schreier diagram of F_0 with respect to F, where F is the pre-image of H in F_0. If we choose T_0 as in Fig. 2(i) then the corresponding Schreier transversal is $\{1, a^{-1}, a^{-1}b\}$; and since there are four edges in Fig. 1(ii) that are not in T_0 we have rank F = 4.

What else can be studied in the graph-theoretical framework? By way of illustration, consider the following result.

PROPOSITION: A group G is a free product if and only if there is a tree on which G acts (i) regularly on the edges but (ii) not transitively on the vertices.

Proof: Assume that G acts with properties (i) and (ii) on the tree T, and let e = xy be a particular edge in T. Property (i) says that, given any edge f in T there is a unique element of G which moves e to f; we label f by the corresponding $g \in G$, so that e is labelled by 1. It follows that every vertex is in the same orbit as (that is, can be moved to) at least one of x or y; but in view of (ii) there are then exactly two vertex orbits, and the end vertices of any edge are in different orbits. Hence, the edges $\neq$ e that meet x (respectively y) are labelled by the non-identity elements of $A = G_x$, (respectively $B = G$

the stabiliser of x (respectively y). We note that $A \cap B = 1$ by (i). A routine, if tedious, argument shows that each edge $\neq$ e is labelled by an alternating product of non-trivial elements from A and B, thus identifying G with the free product $A*B$.

Conversely, given $G = A*B$, construct T as follows: the vertices of T are the cosets of A and B in G, and Ag is joined to Bh if and only if $Ag \cap Bh \neq \phi$. It is clear that G acts on T by right multiplication and that there are two orbits of vertices, so (ii) holds. The standard properties of the free product ensure that T is a tree and that (i) holds also.

A familiar fact follows readily:

COROLLARY: Let H be a subgroup of the free product $G = A*B$. If no conjugate of H meets A or B non-trivially then H is free.

Proof: Let G act on a tree T as in the proposition. For any vertex z of T the stabiliser G_z is conjugate in G to A or to B according as z is in the orbit of A-cosets or the orbit of B-cosets. Thus the hypothesis of the corollary ensures that $H \cap G_z = 1$; in other words, H acts freely on the vertices of T. Since H, being a subgroup of G, also acts freely on the edges of T, we deduce from the earlier theorem that H is free.

Remarks: (a) It is possible to produce similar 'special case' treatments for *free products with amalgamation* and for *HNN groups*; whether this is worth doing depends on, among other things, the amount of time available for the course. It might be argued that if there is time for several special cases then one should treat the general structure theorem of I.5 [6]. However, I feel that students should meet the various constructions separately in a first treatment; the general theorem can follow if there is time - and a desire - for it.

(b) It may be worth mentioning the somewhat surprising fact that the theory of groups acting on (infinite) trees is significant in the study of finite groups; see, for example, Goldschmidt's article [3].

(c) For a more topological account of the Bass-Serre Theory see Cohen's notes [2] or the Scott-Wall article [5].

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Department of Mathematics,
University College,
Galway.

AN INTUITIVE PROOF OF BROUWER'S FIXED POINT THEOREM IN R^2

Clarence C. Morrison[1] *and Martin Stynes*[2]

Fixed point theorems play a major role in general equilibrium theory. Brouwer's theorem is the most basic of these; it states that any continuous function mapping a closed bounded convex set into itself must contain at least one fixed point (i.e., a point that is its own image).

Elementary discussions invariably give an intuitive proof of the theorem for functions of a single variable, as illustrated in Fig. 1. In R^1 a set is convex if and only if it is an interval; thus a continuous mapping of the closed boundary interval $[x_0,x_1]$ into itself can be represented by a curve f.

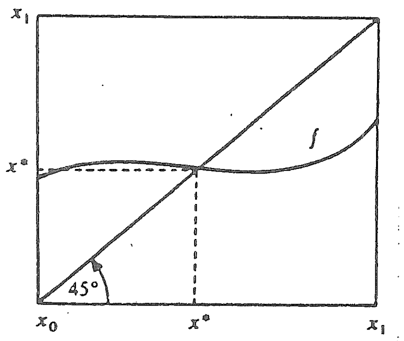

FIGURE 1

Since f connects the left-hand side of the rectangle to the right-hand side of the rectangle, it is intuitively obvious that f must intersect the diagonal of the rectangle at least

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