

PROBLEMS

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The first problem this issue was contributed by Finbarr Holland of University College Cork.

Problem 80.1. Let x_0, x_1, x_2, \dots be a null sequence generated by the recurrence relation

$$(n + 1)(x_{n+1} + x_n) = 1, \quad n = 0, 1, 2, \dots .$$

Prove that the series

$$\sum_{n=0}^{\infty} (-1)^n x_n$$

converges, and determine its sum.

The next problem was suggested by J.P. McCarthy of Cork Institute of Technology.

Problem 80.2. Let

$$S(\sigma) = \sum_{i=1}^n \frac{1}{\sqrt{n^{i+\sigma(i)}}},$$

where σ is a nonidentity permutation of $\{1, 2, \dots, n\}$. Find the maximum of S over all such permutations.

We finish with an easy problem, quoted verbatim from *Lectures and Problems: A Gift to Young Mathematicians*, by V.I. Arnold. The book gives an appealing backstory to the problem, which we will pass on with the solution, in *Bulletin* Number 82.

Problem 80.3. Two volumes of Pushkin, the first and the second, are side-by-side on a bookshelf. The pages of each volume are 2cm thick, and the front and back covers are each 2mm thick. A bookworm has gnawed through (perpendicular to the pages) from the

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first page of volume 1 to the last page of volume 2. How long is the bookworm's track?

SOLUTIONS

Here are solutions to the problems from *Bulletin* Number 78. The first problem was solved by Prithwjit De of Mumbai, India, Henry Ricardo of the Westchester Area Math Circle, New York, USA, the North Kildare Mathematics Problem Club, as well as the proposer, Ángel Plaza of Universidad de Las Palmas de Gran Canaria, Spain. We present a solution, which is equivalent to that of De, Ricardo and Plaza.

Problem 78.1. Given positive real numbers a , b , c , u and v , prove that

$$\frac{a}{bu + cv} + \frac{b}{cu + av} + \frac{c}{au + bv} \geq \frac{3}{u + v}.$$

Solution 78.1. Let L be the expression on the left. Then

$$L = \frac{a^2}{abu + cav} + \frac{b^2}{bcu + abv} + \frac{c^2}{cau + bcv}.$$

By the Cauchy–Schwarz inequality,

$$L((abu + cav) + (bcu + abv) + (cau + bcv)) \geq (a + b + c)^2.$$

Hence

$$\begin{aligned} L(ab + bc + ca)(u + v) &= L((abu + cav) + (bcu + abv) + (cau + bcv)) \\ &\geq (a + b + c)^2 \\ &\geq 3(ab + bc + ca), \end{aligned}$$

where, to obtain the last line, we have expanded the brackets and applied inequalities of the type $a^2 + b^2 \geq 2ab$. Dividing throughout by $(ab + bc + ca)(u + v)$ gives the result we require. \square

The second problem from *Bulletin* Number 78 originated from an exercise found in an Open University complex analysis module. The exercise asks students to prove, in the case $n = 2$, that if p is a polynomial of degree n with n distinct fixed points a_1, \dots, a_n , then at least one of these fixed points a_j satisfies $|p'(a_j)| \geq 1$ (it is a repelling fixed point). Assuming that p is monic, and $n > 1$, we can write

$$p(z) = z + \prod_{i=1}^n (z - a_i),$$

and hence

$$p'(a_j) = 1 + \prod_{\substack{i=1 \\ i \neq j}}^n (a_j - a_i).$$

Problem 78.2 is thereby equivalent to the general form of the student exercise.

The problem can be solved by complex dynamics, arguing that each component of the Fatou set of p can have at most one attracting fixed point, and each such component contains a critical point of p , of which there are at most $n - 1$. However, we favour the solutions submitted (jointly) by Prithwjit De and Finbarr Holland, and also by the North Kildare Mathematics Problem Club. The two solutions were much the same, and we present them here.

Problem 78.2. Let a_1, \dots, a_n be distinct complex numbers. Prove that

$$\left| 1 + \prod_{\substack{i=1 \\ i \neq j}}^n (a_j - a_i) \right| \geq 1$$

for at least one of the integers $j = 1, \dots, n$.

Solution 78.2. The inequality is trivial if $n = 1$, so suppose that $n \geq 2$. Define

$$f(z) = \prod_{i=1}^n (z - a_i).$$

Observe that

$$f'(a_j) = \prod_{\substack{i=1 \\ i \neq j}}^n (a_j - a_i), \quad j = 1, \dots, n.$$

Observe also that the residue of $1/f$ at the simple pole a_j is $1/f'(a_j)$. Now let Γ_R be the circular path of radius R centred at 0, where R is chosen to be greater than $|a_j|$, for $j = 1, \dots, n$. Then, by the Residue Theorem,

$$\frac{1}{2\pi i} \int_{\Gamma_R} f(z) dz = \sum_{j=1}^n \frac{1}{f'(a_j)}.$$

Since $|f(z)| \geq \prod_{i=1}^n (|z| - |a_i|)$, we see that the integral converges to 0 as $R \rightarrow \infty$. Therefore

$$\sum_{j=1}^n \frac{1}{f'(a_j)} = 0.$$

It follows that

$$\sum_{j=1}^n \frac{2\operatorname{Re}(f'(a_j))}{|f'(a_j)|^2} = 0,$$

so

$$\sum_{j=1}^n \frac{|1 + f'(a_j)|^2}{|f'(a_j)|^2} = n + \sum_{j=1}^n \frac{1}{|f'(a_j)|^2}.$$

Consequently $|1 + f'(a_j)| > 1$ for at least one integer j . \square

The third problem was solved by Ángel Plaza, the North Kildare Mathematics Problem Club, and the proposer, Finbarr Holland. All solutions were more or less equivalent. Finbarr points out that the problem is in fact much the same as Problem 11946 from the December 2016 issue of *The American Mathematical Monthly*.

Problem 78.3. Suppose that the continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is twice differentiable on $(0, 1)$ and the second derivative f'' is square integrable on $[0, 1]$. Suppose also that $f(0) + f(1) = 0$. Prove that

$$120 \left| \int_0^1 f(t) dt \right|^2 \leq \int_0^1 |f''(t)|^2 dt,$$

and show that 120 is the best-possible constant in this inequality.

Solution 78.3. Integrating by parts twice, we see that

$$\begin{aligned} \int_0^1 t(1-t)f''(t) dt &= - \int_0^1 (1-2t)f'(t) dt \\ &= -2 \int_0^1 f(t) dt. \end{aligned}$$

Since

$$\int_0^1 t^2(1-t)^2 dt = \frac{1}{30},$$

we can apply the Cauchy–Schwarz inequality, to give

$$\frac{1}{30} \int_0^1 |f''(t)|^2 dt \geq \left| \int_0^1 t(1-t)f''(t) dt \right|^2 = 4 \left| \int_0^1 f(t) dt \right|^2.$$

The result follows on multiplying throughout by 30.

Equality is attained if and only if $f''(t) = kt(1 - t)$, for some constant k , which is so if and only if $f(t) = a(t^4 - 2t^3) + bt + c$, for constants a , b and c . The condition $f(0) + f(1) = 0$ implies that $-a + b + 2c = 0$, so equality is attained for any function of the form

$$f(t) = a(t^4 - 2t^3 + t) - c(2t - 1). \quad \square$$

We invite readers to submit problems and solutions. Please email submissions to imsproblems@gmail.com in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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