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PROBLEMS

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Problems

The first pair of problems were proposed by Professor Tom Moore of Bridgewater State University, USA.

Problem 71.1. For $n = 0, 1, 2, \ldots$, the triangular numbers T_n and Jacobsthal numbers J_n are given by the formulas

$$T_n = \frac{n(n+1)}{2} \quad \text{and} \quad J_n = \frac{2^n - (-1)^n}{3}.$$

(a) Prove that for each integer $n \geqslant 3$ there exist positive integers a, b, and c such that $T_n = T_a + T_b T_c$.
(b) Prove that infinitely many square numbers can be expressed in the form $J_a J_b + J_c J_d$ for positive integers a, b, c, and d.

The next problem was contributed by Finbarr Holland.

Problem 71.2. Prove that for each integer $n \geqslant 3$,

$$\int_0^\infty \frac{x-1}{x^n - 1}\, dx = \frac{\pi}{n \sin(2\pi/n)}.$$

The final problem, proposed by Anthony O'Farrell, is based on an assertion made by the late E. P. Dolženko in a Russian manuscript published in 1963 (see lines 7–8 on page 34 of the English translation in *American Mathematical Society Translations. Series 2. Vol. 97*, Amer. Math. Soc., Providence, RI, 1971, 33–41). The solution is not known to the proposer or to the editor of these problems.

Problem 71.3. Suppose that you remove from a circular disc its intersection with any number of larger circular discs. Is the perimeter of the resulting set necessarily less than or equal to the circumference of the original circular disc?

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Solutions

Here are the solutions to the problems from *Bulletin* Number 69. All three solutions were contributed by the North Kildare Mathematics Problem Club (and each was also solved by the proposer, although the methods of solution differ).

Problem 69.1. Suppose that the matrices A, b, and c are of sizes $n \times n$, $n \times 1$, and $1 \times n$, respectively. Prove that, for all complex numbers z,

$$\det(A - zbc) = \det A - zcA^*b = \det A + z\det\begin{pmatrix} 0 & c \\ b & A \end{pmatrix},$$

where A^* is the adjoint of A (that is, the transpose of the matrix of cofactors of A).

Solution 69.1. The rank of the $n \times n$ matrix bc is less than or equal to 1, which implies that the nullity (the geometric multiplicity of the eigenvalue 0) is greater than or equal to $n - 1$. It follows that the algebraic multiplicity of the eigenvalue 0 is also greater than or equal to $n - 1$. Since the sum of the eigenvalues of bc is equal to its trace cb, we see that the only eigenvalues of bc are cb and 0 (possibly $cb = 0$), and the characteristic polynomial is given by

$$\det(\lambda I - bc) = (\lambda - cb)\lambda^{n-1},$$

where I is the $n \times n$ identity matrix. Evaluating this equation at $\lambda = 1$ gives

$$\det(I - bc) = 1 - cb.$$

Next, replace b by $A^{-1}b$, for some $n \times n$ invertible matrix A, to give

$$\det(I - A^{-1}bc) = 1 - cA^{-1}b.$$

Then multiplying throughout by $\det A$:

$$\det(A - bc) = \det A - cA^*b.$$

Since the collection of invertible $n \times n$ matrices is dense in the space of all $n \times n$ matrices, and both sides of the above equation are continuous in A, we see that the equation is valid for any $n \times n$ matrix A. Finally, we obtain the first of the given identities on replacing b by zb, for a complex number z.

The second identity is obvious, on expanding

$$\det\begin{pmatrix} 0 & c \\ b & A \end{pmatrix}$$

along the first row. □

Problem 69.2. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k} = \zeta(3),$$

where ζ is the Riemann zeta function.

Solution 69.2. Let

$$u_n = \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k}.$$

For $|z| < 1$, we have

$$\begin{aligned} \frac{-\log(1-z)}{1-z} &= \left(\sum_{n=1}^{\infty} \frac{z^n}{n} \right) \cdot \left(\sum_{n=0}^{\infty} z^n \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) z^n \\ &= \sum_{n=1}^{\infty} (n+1)^2 u_n z^n. \end{aligned}$$

Thus, given $0 < x, y < 1$,

$$\sum_{n=1}^{\infty} (n+1)^2 u_n (xy)^n = \frac{-\log(1-xy)}{1-xy}.$$

Integrating once with respect to x and once with respect to y gives

$$\sum_{n=1}^{\infty} u_n (xy)^{n+1} = -\int_0^x \int_0^y \frac{\log(1-st)}{1-st} \, ds \, dt.$$

Since $\sum_{n=1}^{\infty} u_n$ converges, it follows by the Abel–Dirichlet theorem that

$$\sum_{n=1}^{\infty} u_n = -\int_0^1 \int_0^1 \frac{\log(1-st)}{1-st} \, ds \, dt.$$

Substituting $u = st$ and $v = t$, this becomes

$$\begin{aligned} -\int_0^1 \frac{1}{v}\int_0^v \frac{\log(1-u)}{1-u}\,dudv &= \frac{1}{2}\int_0^1 \frac{(\log(1-v))^2}{v}\,dv \\ &= \frac{1}{2}\int_0^1 \frac{(\log(x))^2}{1-x}\,dx \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\int_0^1 x^m(\log(x))^2\,dx. \end{aligned}$$

Integrating by parts,

$$\frac{1}{2}\int_0^1 x^m(\log(x))^2\,dx = -\int_0^1 \frac{x^m\log(x)}{m+1}\,dx = \frac{1}{(m+1)^3}.$$

Therefore

$$\sum_{n=1}^{\infty} u_n = \sum_{m=0}^{\infty}\frac{1}{(m+1)^3} = \zeta(3),$$

as required. □

Problem 69.3. A rectangle is partitioned into finitely many smaller rectangles. Each of these smaller rectangles has a side of integral length. Prove that the larger rectangle also has a side of integral length.

Solution 69.3. We may assume that the larger rectangle R lies in the Cartesian plane and each of its sides is parallel to one of the axes of the plane. Any two rectangles in the partition that meet at more than one point must meet in an interval. It follows that each side of each rectangle in the partition is also parallel to one of the axes of the plane.

Consider now the double integral

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} e^{2\pi i(x+y)}\,dxdy.$$

This integral vanishes if and only if the rectangle with vertices (x_1,y_1), (x_1,y_2), (x_2,y_1), and (x_2,y_2) has a side of integral length. Therefore the integral vanishes when evaluated on each of the rectangles in the partition. It follows that the integral also vanishes on R, so R has a side of integral length. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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