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M. Hayes,
Mathematical Physics Department,
University College,
Belfield,
Dublin 4.

WRITING COMMUTATORS OF GROUP COMMUTATORS AS PRODUCTS OF CUBES

Peter V. Hegarty

Abstract We derive an upper bound for the number of cubes needed to write a commutator of group commutators as a product of cubes.

1. Introduction

If a and b are elements of a group G , we define their commutator $[a, b]$ to be the group element $a^{-1}b^{-1}ab$. It is well known that groups of exponent 3 are metabelian—for a proof see [2], pp. 382-3. Consequently, in the free group F_4 on four generators x, y, z and w , the “commutator of commutators” $[[x, y], [z, w]]$ can be expressed as a product of cubes of elements of F_4 . In the survey article [1], R. Lyndon poses the problem of finding such an expression which contains the smallest possible number of cubes. At this point it is instructive to recall, by way of analogy, the simple and well known fact that, in the free group F_2 on two generators x and y , the commutator $[x, y]$ can be written as a product of 3 squares, but of no fewer:

$$[x, y] = (x^{-1})^2(xy^{-1})^2y^2.$$

Lyndon’s problem, by contrast, seems to be more difficult. In this note, I will show that $[[x, y], [z, w]]$ can be expressed as a product of 85 cubes. This will, I hope, provide a benchmark for future progress on the problem. Following my proof, one could write down an explicit expression, but in the interests of saving space I shall not do so here.

2. Notation

From now on, all work takes place in the free group F_4 on the four generators x, y, z, w . To simplify the presentation of the proof to

follow, let me introduce some unorthodox notation. The symbol π_n will be used to denote a generic product of n cubes of elements of F_4 . It is important to understand that π_n does not denote a group element, but rather a type of group element. Hence, for example, given $a \in F_4$ we will write the equation

$$a = \pi_n \quad (1)$$

to denote the fact that a can be expressed as a product of some (unspecified) n cubes of elements of F_4 . Next, for a and b in F_4 , the equation

$$a = b\pi_n \quad (2)$$

will be written instead of $b^{-1}a = \pi_n$. An elementary but important fact is that, for any $a \in F_4$ and any n , we have

$$a\pi_n = \pi_n a \quad (3)$$

or, in words, right-multiplying a by a product of n cubes is the same as left-multiplying a by some other product of n cubes. The verification of this fact is trivial. More generally, for a_1, \dots, a_k in F_4 and positive integers n_1, \dots, n_k we have

$$\begin{aligned} a_1\pi_{n_1}a_2\pi_{n_2}\dots a_k\pi_{n_k} &= a_1a_2\dots a_k\pi_{n_1+n_2+\dots+n_k} \\ &= \pi_{n_1+n_2+\dots+n_k}a_1a_2\dots a_k. \end{aligned} \quad (4)$$

3. Main Result

We begin with a lemma.

Lemma. For a and b in F_4 , the following hold:

- (i) $[a, b] = [b^{-1}, a]\pi_3$;
- (ii) $[a, b] = [b, a^{-1}]\pi_3$;
- (iii) $[a, b] = [a^{-1}, b^{-1}]\pi_3$.

Proof: For (i), we have

$$\begin{aligned} [a, b^{-1}][a, b] &= a^{-1}ba(b^{-1}a^{-1}b^{-1})ab \\ &= a^{-1}ba(b^{-1}a^{-1})^3aba^2b \\ &= a^{-1}(ba^2)^2b\pi_1 \\ &= a^{-1}(ba^2)^3a^{-2}\pi_1 = \pi_3, \end{aligned}$$

as required. Part (ii) follows from (i) simply by inverting both sides and interchanging the symbols a and b .

For (iii), we have

$$\begin{aligned} [a, b] &= a^{-1}b^{-1}ab = a^{-1}(b^{-1}a)^3a^{-1}ba^{-1}b^2 \\ &= \pi_1(a^{-2}ba^{-1}b^2) = \pi_1(aa^{-3}ba^{-1}b^{-1}b^3) \\ &= \pi_3(aba^{-1}b^{-1}) = \pi_3[a^{-1}, b^{-1}], \end{aligned}$$

as required.

The rest of the paper is devoted to proving the following result.

Theorem. In $F_4 = \langle x, y, z, w \rangle$ we have that $[[x, y], [z, w]] = \pi_{35}$.

Proof: Our strategy is to adapt the proof in [1], pp 382-3, which the reader may profitably consult, that groups of exponent 3 are metabelian. So let us begin.

For a, b and d in F_4 we have

$$\begin{aligned} d^{-1}b^{-1}a^{-1}b^{-1}ada^{-1}d &= d^{-1}(b^{-1}a^{-1})^2a^3(a^{-1}d)^2 \\ &= (d^{-1}abd^{-1}a)\pi_3 \\ &= (d^{-1}ab)^3b^{-1}a^{-1}db^{-1}\pi_3 \\ &= (b^{-1}a^{-1}db^{-1})\pi_4. \end{aligned}$$

So, by equating the first and last terms we get

$$a^{-1}b^{-1}ada^{-1} = (bdb^{-1}a^{-1}db^{-1}d^{-1})\pi_4 \quad (5)$$

Now substitute $bc b^{-1}$ for d in (5) and derive easily that

$$\begin{aligned} c^{-1}a^{-1}b^{-1}abcb^{-1}a^{-1}ba &= (c^{-1}b^2cb^{-2}a^{-1}bcb^{-1}c^{-1}a)\pi_4 \\ &= (c^{-1}b^{-1}cba^{-1}bcb^{-1}c^{-1}a)\pi_6 \end{aligned} \quad (6)$$

Using part (iii) of the lemma, we can obtain from (6) that

$$[c, [b, a]] = [[b, c], a]\pi_9 \quad (7)$$

Now set $u = [[x, y], [z, w]]$ and $g = [z, w]$. Then

$$\begin{aligned} u &= [[x, y], g] = [y, [x, g]]\pi_9 \text{ by (7)} \\ &= [y, [x, [z, w]]]\pi_9 \\ &= [y, [[z, x], w]]\pi_9\pi_9 \text{ by (7)} \\ &= [y, [[z, x], w]]\pi_{27} \\ &= [y, [w, [x, z]]]\pi_3\pi_{27} \text{ by part (ii) of the lemma} \\ &= [y, [w, [x, z]]]\pi_{33} \\ &= [[w, y], [x, z]]\pi_{42} \text{ by (7) again.} \end{aligned}$$

Equating the first and last terms, we have

$$[[x, y], [z, w]] = [[w, y], [x, z]]\pi_{42} \quad (8)$$

Similarly,

$$\begin{aligned} u &= [[x, y], g] = [g, [y, x]]\pi_3 \text{ by part (ii) of the lemma} \\ &= [[y, g], x]\pi_{12} \text{ by (7)} \\ &= [[y, [z, w]], x]\pi_{12} \\ &= [[[z, y], w], x]\pi_9\pi_{12} \text{ by (7)} \\ &= [[[z, y], w], x]\pi_{30} \\ &= [x, [w, [z, y]]]\pi_{33} \text{ by part (ii) of the lemma} \\ &= [[w, x], [z, y]]\pi_{42} \text{ by (7) again.} \end{aligned}$$

Equating the first and the last terms, we have

$$[[x, y], [z, w]] = [[w, x], [z, y]]\pi_{42} \quad (9)$$

Combining (8) and (9) we obtain easily that

$$u = u^{-1}\pi_{84} \Rightarrow u^2 = \pi_{84} = u^3u^{-1} \Rightarrow u = \pi_{85},$$

which completes the proof of the theorem.

Remark Let me elucidate the idea of the proof above. Equation (8) says, informally, that the permutation $x \rightarrow w \rightarrow z$ of the symbols x, y, z, w in the commutator u corresponds to multiplication of u by some 42 cubes. Equation (9) says the same for the permutation $x \rightarrow w \rightarrow y$. The product of these permutations is ' $x \leftrightarrow z, y \leftrightarrow w$ ', which takes u to u^{-1} . Now, though I have not checked it, I would conjecture that any 3-cyclic permutation of x, y, z, w corresponds to multiplication of u by some 42 cubes. An obvious question to ask is whether 42 is best possible. And one may ask the same question for other types of permutations, in particular for transpositions. In this way, it may be possible to improve on the number 85 in our theorem simply by pure luck and without introducing any essentially new ideas. It seems an entirely more complicated matter, however, to obtain optimal results.

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References

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P. V. Hegarty
Department of Mathematics,
Princeton University,
Princeton, NJ 08544,
USA.