DAVID HILBERT AND THE THEORY OF ALGEBRAIC INVARIANTS

David W. Lewis

1. Introduction

The theory of algebraic invariants was at the forefront of mathematics in the latter half of the 19-th century. It attracted the interest of many top-class mathematicians. For example Cayley and Sylvester in England were known as the "Invariant Twins", and when Salmon in Dublin made useful contributions to the subject the trio were christened by Hermite as the "Invariant Trinity". Another Irish link with invariant theory is provided by George Boole who spent much of his working life in Cork. In 1841 Boole wrote a paper [1] which is often regarded as the beginnings of invariant theory, and in 1845 he wrote another paper on the subject but seemed to do nothing further on invariants. (Admittedly Boole was still in England when he wrote these papers. He moved to Ireland to become Professor of Mathematics at Queen's College, Cork in 1849). See [10] for an excellent account of the life and work of Boole. The Italian mathematician Faà di Bruno wrote a book on invariant theory which was highly regarded by Hilbert. In Germany the first mathematician to draw attention to the theory of invariants was Aronhold. He was followed by Clebsch and Gordan who worked extensively on the subject and developed symbolic calculation in invariant theory. Indeed Gordan was known as the "King of Invariants" and apparently would talk interminably about invariant theory to anyone who was willing to listen. (The names of Clebsch and Gordan will be familiar to students of quantum mechanics via the Clebsch-Gordan series and Clebsch-Gordan coefficients. The Clebsch-Gordan series played an important role in their theory of invariants of binary forms. See Weyl

[19].) Their work involved massive calculations. According to [3], there are papers of Gordan where twenty pages of formulae are not interrupted by a single text word, and it is alleged that Gordan often wrote only the formulae in his papers, the text being added later by friends.

Although invariant theory was a piece of pure mathematics, attempts were made to make use of invariant theory in other disciplines. For example, Sylvester in 1878, and later Gordan and Alexejeff, tried to apply invariant theory to chemistry, in connection with chemical valency. A brief account of this so-called "chemico-algebraic theory" appears in [5, pp.366-368]. In the period from 1885 to 1893 David Hilbert demolished the old-style invariant theory by solving, in a novel and unexpected way, the central finiteness problem of invariant theory. After Hilbert's work, many people thought that invariant theory was a dead subject. However it has refused to lie down and has resurrected itself on quite a few occasions in the 20th century. Indeed, to quote from the 1984 survey article by Kung and Rota [9], "the theory of invariants, pronounced dead at the turn of the century, is once again at the forefront of mathematics". Today, invariant theory is alive and well and the subjects of commutative algebra, algebraic geometry, representation theory , and combinatorics each owe an important debt to invariant theory.

2. David Hilbert

David Hilbert was born in 1862 in Königsberg, then part of East Prussia but renamed Kaliningrad after the Second World War and now a part of Russia. Königsberg has a long intellectual tradition, especially in mathematics and philosophy. (Mathematicians will all know of the famous "Königsberg bridge problem" solved by Euler in the 18th century. The philosopher Kant was one of the city's most famous sons. Clebsch was also born in Königsberg and attended the university there.) Hilbert went to university in Königsberg where he became a close friend of fellow student Hermann Minkowski, this friendship lasting until Minkowski's early death in 1909. After spending several very productive years lecturing at Königsberg, during which time he did all of his important

work in invariant theory, he was offered and accepted in 1895 a position at Göttingen. The mathematics department there, with Felix Klein as chairman, was possibly the most prestigious in Germany at that time. Hilbert spent the rest of his life in Göttingen and died there in 1943.

Hilbert has been described as "the last of the great universalists". Over his long career he made vital contributions to large and diverse areas of mathematics. One might say that he led mathematics out of the 19th century and into the 20th century. His famous list of unsolved problems at the International Congress of Mathematicians in Paris in 1900 pointed the way forward and profoundly influenced the direction of mathematical research in this century. His method of work led to great advances both in technical results and in the way in which mathematicians think about mathematics. Hilbert tended to concentrate almost exclusively on one particular area of mathematical research for a period of years and then move on to a different branch. His research work, according to [17], was roughly as follows:

1885-1893 - Invariant Theory
1893-1898 - Number Theory
1898-1902 - Foundations of Geometry and of Mathematics in general
1902-1912 - Integral Equations
1912-1922 - Mathematical Physics

The Japanese number theorist Takagi visited Hilbert at Göttingen in 1902 but Hilbert is reported to only have been being interested in talking about integral equations (the work of Takagi and of Hilbert forms the beginnings of class field theory). See [13, p.86]. There were exceptions to the list above. For example, in 1909 Hilbert successfully solved Waring's problem, a problem outstanding since 1770 about expressing a natural number as a sum of n-th powers. He produced this solution just at the time his friend Minkowski was dying of appendicitis and unfortunately Minkowski was unable to attend the seminar by Hilbert in which he described his solution to the problem. Also in 1899 Hilbert managed to resuscitate Dirichlet's Principle concerning the solution of boundary value problems, this being totally unrelated to

the main research work he was pursuing at this period. It was in the period on foundations of geometry that he made his famous pronouncement about the axiomatic method-"One must be able to say at all times-instead of points, straight lines and planes - tables, chairs, and beer mugs".

From 1912 on he worked on the idea of axiomatizing physics– this having been proposed as his 6th problem at the 1900 Paris congress. "Physics is much too hard for physicists", he said. He did not have much success however, the axiomatic method not seeming to be suitable for physics.

For a full account of the life of Hilbert the reader should refer to the book of Constance Reid, [13]. See also the book of Fang, [3].

3. Hilbert's 1897 lectures

In 1897 David Hilbert gave an introductory course of lectures on the theory of algebraic invariants at the University of Göttingen. These lectures, or rather a modern English translation by Reinhard C. Laubenbacher of the lecture notes, handwritten by Hilbert's student Sophus Marxsen, have recently been published by the Cambridge University Press, [6]. They provide a fascinating view of invariant theory and a glimpse of what it must have been like to have studied with Hilbert at Göttingen at that time. The course consisted of 51 lectures starting on 26 April 1897 and ending on 6 August 1897, 3 lectures per week for 17 weeks. Sophus Marxsen ended up with 527 pages of handwritten notes. As a lecturer Hilbert was inspiring but he sometimes ran into difficulty in a lecture because he had not prepared all the technical details. (This was in sharp contrast to his Göttingen colleague Felix Klein who is reported to always have prepared everything in meticulous detail. Klein was older and more famous than Hilbert at that time, although nowadays he is perhaps best remembered for his bottle, the "Klein bottle" being the famous one-sided surface loved by all topologists.) The year 1897 was an appropriate time for Hilbert to give an expository course on invariant theory because in two papers [7], [8], in 1890 and 1893 he had solved the major problems in invariant theory. Thus he was able in these lec-

tures to incorporate the work of his predecessors and his own new and revolutionary approach to the subject. For local interest we should remark that in his first lecture Hilbert refers to the book of Salmon [14], Modern Higher Algebra (fourth edition), Dublin 1885, as one of the best introductions to the subject of invariant theory.

An *m-ary n-form* ϕ is a homogeneous polynomial of degree n in m variables. (For $n = 2$, this is a quadratic form.) If we write $x_1, x_2, \ldots, x_m$ for the variables, then we may write

$$\phi = \sum a_{i_1 i_2 \ldots i_m} x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m}$$

for some suitable constants $a_{i_1 i_2 \ldots i_m}$. In Hilbert's lectures these constants are allowed to be complex numbers.

For $m = 2$, the form is called a *binary* form, for $m = 3$ a *ternary* form etc. The *degree* of the term $a_{i_1 i_2 \ldots i_m} x_1^{i_1} x_2^{i_2} \ldots x_m^{i_m}$ is $i_1 + i_2 + \cdots + i_m$ (All terms of the form will have the same degree because the form is a homogeneous polynomial.)

Suppose we make a linear change of variables from $x_1, x_2, \ldots, x_m$ to $x_1', x_2', \ldots, x_m'$, i.e we write $x = Px'$, where $x = (x_i)$, $x' = (x_i')$ are column vectors and $P = (p_{ij})$ is an $m \times m$ matrix. Then the form may be written in terms of the new variables x_i' with new coefficients $a'_{i_1 i_2 \ldots i_m}$. The determinant of the matrix P is denoted δ and is called the *transformation determinant.*

Hilbert, in his lectures, limits himself to binary forms but says that generalizing to m-ary forms poses no difficulties in most cases. He writes a general binary form ϕ in the manner

$$\phi(x_1, x_2) = \sum_{i=0}^{n} \binom{n}{i} a_i x_1^i x_2^{n-i}.$$

(He always uses the word "coefficients" to mean the a_i when the form is written in this way, i.e. not multiplied by the binomial coefficients!)

An *invariant* of the form ϕ above is a polynomial function $I(a_0, a_1, \ldots, a_n)$ of the coefficients of ϕ which changes only by a

factor equal to a power of the transformation determinant δ when one makes a linear transformation of the variables, i.e.

$$I(a_0', a_1', \ldots, a_n') = \delta^p I(a_0, a_1, \ldots, a_n)$$

for some natural number p. Here a_0', $a_1', \ldots, a_n'$ are the coefficients of ϕ after a linear change of variables given by the matrix P. It can be shown by elementary considerations that I must necessarily be homogeneous of degree g where $ng = 2p$. We illustrate by a couple of examples.

Example 1

$\phi = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$ is a binary form of degree 2. $I_1 = a_0 a_2 - a_1^2$ is an invariant of ϕ. (Those familiar with quadratic forms will recognize ϕ as a quadratic form of dimension 2 and I as the discriminant of this form.)

Example 2

$\phi = a_0 x_1^4 + 4a_1 x_1^3 x_2 + 6a_2 x_1^2 x_2^2 + 4a_3 x_1 x_2^3 + a_4 x_2^4$ is a binary form of degree 4.

$I_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2$ is an invariant of ϕ.

$I_3 = a_0 a_2 a_4 - a_0 a_3^2 - a_1^2 a_4 + 2a_1 a_2 a_3 - a_2^3$ is also an invariant of ϕ.

Observe that I_1 in example 1 and I_2 in example 2 are each homogeneous of degree 2 and I_3 in example 2 is homogeneous of degree 3.

Hilbert proceeds in the first half of these lectures to characterize those polynomials which are invariants by utilizing the operator D defined as follows:

$$D = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + \cdots + n a_{n-1} \frac{\partial}{\partial a_n}.$$

An invariant I is shown necessarily to be a homogeneous polynomial and it must satisfy $DI = 0$. Also it is shown that an invariant I must be an isobaric function of $a_0, a_1, \ldots, a_n$. (A polynomial in $a_0, a_1, \ldots, a_n$ is said to be *isobaric* if each term has the same weight, where the weight of a term $a_0^{\nu_0} a_1^{\nu_1} a_2^{\nu_2} \ldots a_n^{\nu_n}$ is

$$\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + n\nu_n.)$$

Hilbert shows that each isobaric homogeneous polynomial in a_0, $a_1, \ldots, a_n$ of degree g and weight p, where $ng = 2p$, is an invariant whenever $DI = 0$. He also calculates the number of invariants of given degree g for a form ϕ. It turns out that I_1 in example 1 above is the only invariant of a binary quadratic form and that I_2 and I_3 of example 2 are the only invariants of a binary form of degree 4. He also discusses the notion of a covariant, of which invariants are a special case, but we omit discussion of these in this short article.

In lecture 24, Hilbert introduces the idea of simultaneous invariants. Here one begins with an arbitrary set of base forms, each in the same number of variables but not necessarily of the same degree, rather than a single form. One defines an invariant of the set to be a polynomial in the set of coefficients of all the base forms which changes only by a power of the transformation determinant when the same linear transformation is applied simultaneously to all of the base forms. For example, suppose we have binary forms

$$\phi_1(x_1, x_2) = \sum_{i=0}^{n} \binom{n}{i} a_i x_1^i x_2^{n-i}$$

$$\phi_2(x_1, x_2) = \sum_{i=0}^{m} \binom{m}{i} b_i x_1^i x_2^{m-i}.$$

Then a simultaneous invariant for the pair ϕ_1, ϕ_2 under a linear transformation changing the a_i, b_i to a_i', b_i' is a polynomial I in $n + m + 2$ variables such that

$$I(a_0', \ldots, a_n', b_0', \ldots, b_m') = \delta^p I(a_0, \ldots, a_n, b_0, \ldots, b_m)$$

for some natural number p, where δ is the transformation determinant. (Note that any invariant of ϕ_1 alone yields a simultaneous invariant by viewing it as of degree zero in the b_i, and similarly for an invariant of ϕ_2 alone.)

Example 3
Consider the two binary cubic forms

$$\phi_1(x_1, x_2) = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3$$
$$\phi_2(x_1, x_2) = b_0 x_1^3 + 3b_1 x_1^2 x_2 + 3b_2 x_1 x_2^2 + b_3 x_2^3.$$

One may check that $I = a_0 b_3 - 3a_1 b_2 + 3a_2 b_1 - a_3 b_0$ is a simultaneous invariant of these two cubic forms.

Starting with an arbitrary system of base forms, the simultaneous invariants of the system can in general be an infinite set. It was Cayley who first conjectured that any system of base forms has an invariant set which is finitely generated, i.e. there is a finite subset $I_1, I_2, \ldots, I_k$ of the invariant set such that each element of the invariant set is a polynomial in $I_1, I_2, \ldots, I_k$. However, Cayley soon began to doubt the validity of his conjecture and, in an 1856 memoir, he incorrectly claimed that the fundamental system of invariants is infinite for forms of degree more than six. His mistake arose from wrongly taking certain syzygies to be independent. (See below for more about syzygies.) Gordan, via cumbersome calculations using the symbolic method, succeeded in proving the finiteness theorem for an arbitrary system of binary base forms. This achievement in 1868 was what gained Gordan his title of "King of Invariants". However attempts by Gordan himself and others to prove finiteness for base forms of higher degree were unsuccessful. The finiteness problem, i.e. the proof of Cayley's conjecture for an arbitrary system of base forms of any degree, had become the main problem of invariant theory by the time Hilbert came on the scene. (The earlier stages of invariant theory had been concerned with finding the laws governing the structure of invariants and then with the enumeration and production of invariants systematically.) Hilbert solved the finiteness problem by realizing that one only needs to prove the *existence* of a finite basis (i.e. generating set) for the invariants. It was not necessary to construct a basis explicitly, which is what Gordan and others had tried to do. Hilbert's solution when it appeared in 1890, [7], caused consternation amongst mathematicians. His "existence theorem" was not accepted by some of them as being a solution

at all. Gordan commented about the proof, "Das ist nicht Mathematik. Das ist Theologie". Hilbert did indeed continue to work on invariant theory and in [8] he gave an essentially constructive and algorithmic method for obtaining a finite basis.

In the second part of his Göttingen lectures, (lecture 34 onwards), he begins by proving the finiteness theorem for an arbitrary system of *binary* forms. His proof uses a technique called representation by root differences which involves the elementary symmetric functions. It does not generalize to systems of forms of degree greater than two. A key lemma used in this proof asserts that a system of linear equations with coefficients in the natural numbers has a finite number of non-negative solutions which generate all the other non-negative solutions. This lemma is foundational nowadays in the theory of integer programming. See [15]. Hilbert proceeds (in lectures 34-36) to prove his general finiteness theorem as in his 1890 paper, using the key result known nowadays as the Hilbert Basis Theorem for polynomial ideals together with Cayley's Ω-process. The Ω-process is a differentiation process which behaves like a kind averaging and when applied repeatedly to a polynomial it yields an invariant. His Basis Theorem yields a finite set $I_1, I_2, \ldots, I_k$ such that any invariant I is expressible in the form

$$I = F_1 I_1 + F_2 I_2 + \cdots + F_k I_k$$

for some forms $F_1, F_2, \ldots, F_k$. Applying Ω to each of the F_i yields invariants G_i such that we can write

$$I = G_1 I_1 + G_2 I_2 + \cdots + G_k I_k$$

and each G_i clearly has degree less than the degree of I, since each F_i has degree at least one. By expressing each G_i in terms of the set $I_1, I_2, \ldots, I_k$ and repeating as necessary, we can eventually write I as a polynomial in the set $I_1, I_2, \ldots, I_k$. The remainder of the lectures are based on Hilbert's 1893 paper, [8], where he gives his algorithmic method for constructing a finite basis. From a modern perspective there are two highly significant theorems

contained there, although their full importance and application was not apparent then. In lecture 39 he gives the theorem now known as the Hilbert Nullstellensatz, although he refers to [8] for a full proof. This theorem concerning the zero sets of families of polynomials is basic and fundamental for modern commutative algebra and algebraic geometry. Lecture 47 describes the result usually known now as Hilbert's Syzygy Theorem. The set $I_1, I_2, \ldots, I_k$ is not likely to be linearly independent, i.e. there will be a set of relations between them. This relation set also must have a finite basis $R_1, R_2, \ldots, R_h$ by the finiteness theorem. There may well be relations amongst the relations, i.e. expressions of the form

$$S_1 R_1 + S_2 R_2 + \cdots + S_h R_h = 0.$$

Such an expression is called a syzygy of the first order. These syzygies again form an ideal to which the finiteness theorem applies and a finite basis exists. Any relation for this basis is a syzygy of the second order. It may seem that this process can be repeated *ad infinitum*, but Hilbert's Syzygy Theorem says that the chain of syzygies breaks off after finitely many steps. In the last few lectures Hilbert outlines some applications of invariant theory to geometry and discusses possible generalizations of invariant theory.

4. The view from the end of the 20th century

In modern terms we may describe invariant theory as being concerned with the linear action of a group G on a K-vector space V for some field K. Writing $K[V]$ for the ring of all polynomial functions on V, the basic problem is to describe the subring $K[V]^G$ of all polynomials invariant under the action of the group G. In particular, we may ask whether $K[V]^G$ is finitely generated as a K-algebra and, if so, find an algorithm for determining a set of generators. In the classical case described above we have $K = \mathbf{C}$, the complex numbers, $G = \mathrm{GL}_m(\mathbf{C})$, the group of all invertible $m \times m$ matrices with complex entries, V an m-dimensional vector space over K, and $K[V]$ the ring of all homogeneous polynomials in m variables. Hilbert proved that $K[V]^G$ is finitely generated as a K-algebra in the classical case. Hilbert's 14th problem at the

1900 congress asked whether this finiteness theorem remains true if G is an arbitrary subgroup of $\mathrm{GL}_m(\mathbf{C})$. It remained an open problem until 1959 when Nagata [12] answered it in the negative by producing an example of a group G with $K[V]^G$ not finitely generated.

Three of Hilbert's results in the above lectures have turned out to have tremendous significance and importance and we will describe now how they fit into 20th century algebra. The Hilbert Basis Theorem is now usually stated in the form that the polynomial ring $K[x_1, x_2, \ldots, x_n]$ is a noetherian ring. A ring is said to be *noetherian* if every ascending chain of ideals terminates. It is not hard to prove that this ascending chain condition for the polynomial ring is equivalent to the ideals being finitely generated. The name noetherian is after Emmy Noether who, in the 1920's and 1930's, was the main influence in the development of modern abstract algebra. It is curious that Emmy Noether began her career as a student of Paul Gordan at Erlangen, writing a thesis in 1907 on invariant theory. Gordan was still doing very computational invariant theory. Noether later referred to invariant theory as a "jungle of formulae" (formelngestrüpp) and one may speculate that it was her distaste for this kind of mathematics which led her to develop the conceptual approach of modern abstract algebra.

Hilbert's Nullstellensatz is now usually regarded as the foundation of algebraic geometry, yielding the correspondence between geometric objects (varieties) and algebraic objects (co-ordinate rings), although we have seen that this was not the purpose for which Hilbert originally developed it.

Hilbert's Syzygy Theorem is now stated as a result in homological algebra, saying that the polynomial ring $\mathbf{C}[x_1, x_2, \ldots, x_n]$ has finite global dimension (in fact dimension n), i.e. every module over this polynomial ring admits a finite free resolution of length at most n.

We finish with a few words about how invariant theory has developed in the 20th century, although this author claims no great expertise in modern invariant theory. Weyl [18] developed invariant theory for all the classical Lie groups and linked it with

representation theory. Mumford [11] developed a geometric invariant theory. The survey by Kung and Rota, [9], describes invariant theory from the viewpoint of modern combinatorial theory. The books by Springer [14] and by Dieudonné and Carrell [2] are further modern references on the subject. As a final remark, we note that the "death" of invariant theory has even attracted the interest of a sociologist! See [4].

References

[1] G. Boole, *Exposition of a general theory of linear transformations, I and II* Cambridge Math. Journal **3** (1842), 1-20 and 106-119.

[2] J. Dieudonné and J. B. Carrell, Invariant Theory, old and new. Academic Press: New York and London, 1971.

[3] J. Fang, Hilbert: Towards a Philosophy of Modern Mathematics II. Paideia Press: New York, 1970.

[4] C. S. Fisher, *The death of a mathematical theory. A study in the sociology of knowledge*, Arch. Hist. of Exact Sciences **3** (1966), 137-159.

[5] J. H. Grace and A. Young, The Algebra of Invariants. Cambridge University Press: Cambridge, 1903.

[6] David Hilbert, Theory of Algebraic Invariants, (English translation). Cambridge University Press: Cambridge, 1993.

[7] David Hilbert, *Über die Theorie der algebraischen Formen*, Math. Ann. **36** (1890), 473-531.

[8] David Hilbert, *Über die vollen Invariantensysteme*, Math. Ann. **42** (1893), 313-370.

[9] J. P. S. Kung and G-C. Rota, *The invariant theory of binary forms*, Bull. Amer. Math. Soc. **10** (1984), 27-85.

[10] Desmond MacHale, George Boole. Boole Press: Dublin, 1985.

[11] D. Mumford, Geometric Invariant Theory. Springer-Verlag: Berlin-New York, 1965.

[12] M. Nagata, *On the 14th problem of Hilbert*, Amer. J. Math. **81** (1959), 766-772.

[13] Constance Reid, Hilbert. Springer-Verlag: Berlin-Heidelberg-New York, 1970.

[14] G. Salmon, Lessons Introductory to the Modern Higher Algebra (fourth edition). Hodges, Figgis: Dublin, 1885.

[15] A. Schrijver, Theory of Linear and Integer Programming. Wiley-Interscience: Chichester, 1986.

[16] T. A. Springer, Invariant Theory. Lecture Notes in Mathematics 585. Springer-Verlag: Berlin-Heidelberg-New York, 1977.

[17] Hermann Weyl, *David Hilbert and his mathematical work*, Bull. Amer. Math. Soc. **50** (1944), 612-654.

[18] Hermann Weyl, The Classical Groups, Their Invariants and Representations. Princeton University Press: New Jersey, 1939.

[19] Hermann Weyl, The Theory of Groups and Quantum Mechanics, (English translation). Methuen: London, 1931.

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D. W. Lewis,
Department of Mathematics,
University College,
Belfield,
Dublin 4.

SYLOW'S PROOF OF SYLOW'S THEOREM

Rod Gow

1. Introduction

While looking through some early volumes of *Mathematische Annalen*, we came across a paper with the following title:

Théorèmes sur les groupes de substitutions.

Par M. L. Sylow à Frederikshald en Norvege.

This was, of course, the paper containing Ludwig Sylow's fundamental contribution to group theory, [9]. We thought it might be interesting to see how Sylow actually proved his theorem and then to comment briefly on some later proofs and earlier work. It is likely that there have been prior discussions of the history of Sylow's theorem in the literature and we apologize for failing to acknowledge any relevant publications. (Our excuse is that the UCD library is badly stocked with periodicals on the history of science.)

Sylow's starting point is as follows: *On sait que si l'ordre d'un groupe de substitutions est divisible par le nombre premier n, le groupe contient toujours une substitution d'ordre n.* (The notation of Sylow is a little wayward to modern tastes. His prime is denoted by n, rather than the traditional p. Later in the paper, the expression $np+1$ appears as the number of Sylow subgroups, but p denotes merely some non-negative integer. In virtually all later literature relating to the proof of Sylow's theorem and earlier literature on Cauchy's theorem that we have seen, the prime is represented by p. We shall follow standard practice and denote our prime by p in this exposition, except when enunciating Sylow's