ANALYSIS AND TOPOLOGY IN MATHEMATICAL ECONOMICS

Alan F. Beardon

Abstract This article is an expanded set of notes of a lecture given at University College, Cork in April 1993. Mathematical economics provides a fertile source of applications of general topology, and we illustrate this here, as well as discussing some topics which warrant further study.

1. The consumer

We wish to model the behaviour of a consumer faced with the task of buying quantities $x_1, \ldots, x_n$ of n *goods* $G_1, \ldots, G_n$. The *bundle* of goods is $x = (x_1, \ldots, x_n)$ and, to avoid boundary conditions, we usually assume that x lies in the open set

$$\Omega = \{ x \in \mathbf{R}^n : x_j > 0, j = 1, \ldots, n \}.$$

The good G_j has a *unit price* p_j, the *price vector* p is $(p_1, \ldots, p_n)$, and the *cost* of the bundle x is the scalar product $p.x$. Given two bundles x and y, the consumer is assumed to have a (weak) preference for one of them and, formally, this is described as follows.

Definition A *weak preference relation* is a binary relation $\succeq$ on Ω which is

(1) *complete* (either $x \succeq y$ or $y \succeq x$; in particular, $x \succeq x$), and
(2) *transitive* ($x \succeq y$ and $y \succeq z$ implies $x \succeq z$).

We make the natural definitions (i) $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$ (the consumer is then said to be *indifferent* between x and y), and (ii) $x \succ y$ if and only if $x \succeq y$ but not $x \sim y$ (the consumer then has a *strict preference* for x). Of course, we assume that each good is desirable, so that if the bundle x contains as much of each good as bundle y, and more of some good, then

$x \succ y$. Finally, we allow ourselves to use $x \prec y$ for $y \succ x$ and similarly for $\preceq$.

It is easy to see that $\sim$ is an equivalence relation. The $\sim$equivalence class of x is denoted by $I(x)$ and is called the *indifference class* of x. In general, each $I(x)$ is an $(n-1)$-dimensional manifold (for example, a curve if $n = 2$) and these sets play a basic role in any discussion of the standard problems in economics. We may ask, for example, what is the most preferred bundle that can be purchased with a fixed sum of money; do we always buy less of G_j when p_j increases (the answer is 'no'), and so on. For a general account of these ideas and those in the next section, see [1], [2], [8] and [10].

2. Utility functions

The simplest way to construct a preference relation $\succeq$ on Ω is to take a real-valued function $u(x_1, \ldots, x_n)$ that is strictly increasing in each variable x_j, and then define $x \succ y$ if and only if $u(x) > u(y)$. With this, the indifference classes are the level sets of u, and we say that u is a *utility function representing* $\succeq$. As a (popular) example, we mention the Cobb-Douglas utility function given by

$$u(x_1, \ldots, x_n) = x_1^{a_1} \ldots x_n^{a_n}, \quad a_j > 0.$$

It should be noted that we do not attach any significance to the numerical value of $u(x)$, but only to the relative values of $u(x)$ and $u(y)$. It follows that if u is a utility function representing $\succeq$, then so is $h(u(x))$ for any strictly increasing real function h.

A significant part of the theory is devoted to the problem of when (or how) can we represent a given preference relation $\succeq$ by a utility function. This is not a trivial question for, as the next example shows, *such a representation is not always possible.*

Example: the lexicographic ordering Take $n = 2$ and define $(x_1, x_2) \succeq (y_1, y_2)$ if and only if either $x_1 > y_1$, or both $x_1 = y_1$ and $x_2 \geq y_2$. Note that in this case $I(x) = \{ x \}$, a single point. To see that this relation cannot be represented by a utility function, we simply observe that if this were possible, each vertical line in Ω would map by u into an open interval, and these intervals would constitute an uncountable number of disjoint non-empty open intervals in $\mathbf{R}$. As this cannot be done, the function u cannot exist.

It is natural to attempt to represent a preference relation by a *continuous* utility function u, and if $\succeq$ can be so represented then, for each y in Ω, the set

$$\begin{aligned} A(y) &= \{\, x \in \Omega : x \succ y \,\} = \{\, x \in \Omega : u(x) > u(y) \,\} \\ &= u^{-1}(u(y), +\infty) \end{aligned}$$

is open, as is $B(y) = \{\, x \in \Omega : y \succ x \,\}$. In fact, these conditions are sufficient.

Theorem 2.1. *A preference relation $\succeq$ on Ω can be represented by a continuous utility function if and only if for all y in Ω, the sets $A(y)$ and $B(y)$ are open.*

Proof: We have to construct a continuous utility function given that all the sets $A(y)$ and $B(y)$ are open. Let D be the 'diagonal' in Ω (given by $x_1 = x_2 = \cdots = x_n$) and consider the bundle y. It is not hard to see that $I(y)$ meets D (otherwise $A(y)$ and $B(y)$ disconnect D) and, as the goods are desirable, $I(y) \cap D$ must be a single point, say y_D. We now define $u : \Omega \to \mathbf{R}$ by $u(y) = \|y_D\|$, and, because the sets $A(y)$ and $B(y)$ are open, it is easy to show that u is continuous. This result is a simplified version of the more general result in the fundamental paper [21].

The construction of the map $y \mapsto y_D$ applies not just to D but to any ray from the origin, and this really means that, in the circumstances described in Theorem 2.1, each indifference class is radially homeomorphic to that part of the unit sphere lying in Ω, and so is an $(n-1)$-dimensional manifold.

3. Abstract preference relations

We now turn to discuss a preference relation $\preceq$ defined on an arbitrary non-empty set X (such circumstances are of interest to some economists). The definition remains valid, but we lose the concept of having 'more' of a good, and the topology on X (if there is one) may be quite bizarre. However, the induced indifference relation $\sim$ is still an equivalence relation, and the problem of representing $\preceq$ by a utility function remains. We say that the pair $(X, \preceq)$ is a *commodity space.*

The question of representation of $\preceq$ by a *continuous* utility function implies the existence of a topology on X, and even if X comes equipped with a topology (for example, the Euclidean topology), there is no reason at all to suppose that this topology should be related in any intrinsic way to $\preceq$. On the other hand, there *is* a natural topology on X, namely the *order topology* $\mathcal{T}_O$ generated by the intervals $\{\, x : x \succ y \,\}$ and $\{\, x : y \succ x \,\}$, and this topology is obviously equivalent to the preference relation used to define it. Note that the condition given in Theorem 2.1 can now be rephrased as the set-theoretic inclusion $\mathcal{T}_O \subset \mathcal{E}$, where $\mathcal{E}$ is the Euclidean topology on Ω. This type of inclusion arises frequently, and for good reason. The natural question concerning continuity is continuity with respect to the order topology $\mathcal{T}_O$, and if u is continuous with respect to $\mathcal{T}_O$, and if $\mathcal{T}_O \subset \mathcal{T}$, then u is also continuous with respect to $\mathcal{T}$. For more details, see [2], [9], [10] and [14].

We mention, in passing, that if $\mathcal{S}$ is the closed unit square in $\mathbf{R}^2$ ordered lexicographically, then the topological space $(\mathcal{S}, \mathcal{T}_O)$ is an example (different from the usual $\sin(1/x)$ curve) of a space that is connected but not arcwise connected.

4. Existence of utility functions

If we now consider the commodity space $(X, \preceq)$ with the order topology, the quotient space $X/\sim$ is linearly ordered in a natural way and the order topology on this is indeed the quotient topology. Abstractly, then, the existence of a utility function is equivalent to the quotient space $X/\sim$ being order-isomorphic to a subset of $\mathbf{R}$, and there are various results of this type available. For example, we have (see [12])

Theorem 4.1. *An ordered set is order-isomorphic to a subset of $\mathbf{R}$ if and only if it has a countable dense subset (in the order topology), and has only countably many pairs x and y such that $x \prec y$ and $(x, y) \cap X = \emptyset$.*

We call such a pair $\{\, x, y \,\}$ a *jump*. If a linearly ordered set X

has a jump $\{x, y\}$, then the disjoint open intervals $(-\infty, y)$ and $(x, +\infty)$ disconnect X, so we have (see [11] and [12])

Corollary 4.2. *If a commodity space $(X, \preceq)$ is connected in its order topology, or in any larger topology, and if it has a countable dense subset, then $\preceq$ can be represented by a utility function.*

Another existence result ([2], [9], [10] and [19]) is

Theorem 4.3. *Let $(X, \preceq)$ be a commodity space. Then $\preceq$ can be represented by a utility function if and only if the order topology $\mathcal{T}_O$ is second countable.*

Proof: It is clear that if a utility function exists, then $\mathcal{T}_O$ is second countable (because $\mathbf{R}$ is). Now let $O_1, O_2, \ldots$ be a countable basis for $\mathcal{T}_O$. Given x, define

$$N(x) = \{n : O_n \subset (-\infty, x)\}, \quad v(x) = \sum_{n \in N(x)} \frac{1}{2^n}.$$

It is clear that $v(x)$ is weakly increasing with preferences. If $x \prec y$, then $(-\infty, x)$ is a *proper* subset of $(-\infty, y)$ (because the latter set contains x) so that $N(x)$, and hence $v(x)$, is strictly increasing with preferences and v is the required utility function. This completes the proof.

Yet another approach is to mimic the idea in the proof of Theorem 2.1. Suppose for the moment, that a topological space $(X, \mathcal{T})$ is arcwise connected, and that it supports a preference relation $\preceq$ with respect to which there is a maximally preferable point z and a minimally preferable point w. Suppose also that $\mathcal{T}_O \subset \mathcal{T}$. Join z and w by a curve γ in X. As in the proof of Theorem 2.1, every indifference class meets γ and provided that we can construct a utility function on γ, we can extend this to a utility function on X. With a little more work these ideas lead to (see [17])

Theorem 4.4. *Let $(X, \mathcal{T})$ be an arcwise connected topological space, and let $\preceq$ be a preference relation on X such that $\mathcal{T}_O \subset \mathcal{T}$. If X contains a countable subset X_0 such that for all x in X there are points a and b in X_0 such that $a \preceq x \preceq b$, then $\preceq$ can be represented by a utility function.*

Finally, we observe that each preference relation $\succeq$ on X is a subset of $X \times X$, and that this space carries the product topology. As the class of all subsets of a topological space also carries a natural (product) topology, it follows that *there is a natural topology on the space of all preference relations.* With this, we can begin to discuss more sophisticated results (and problems): for example, under certain circumstances, we can find utility functions which are *jointly continuous* with respect to both x and $\preceq$.

5. The non-existence of utility functions

Preference relations which cannot be represented by a utility function are hard to find; indeed, the lexicographic order (or some variation of it) seems to be the only explicit example that is known. A more sophisticated (and implicit) example is that of the 'long-line'.

Example: the long-line A linearly ordered set X is *well-ordered* by $<$ if every non-empty subset X_0 of X has a smallest element, and (assuming the Axiom of Choice) every set can be well-ordered. Let A be any well-ordered, uncountable set ordered, say, by $<$. Now construct another well-ordered set B as follows:
Case 1: if, for each a in A, $(-\infty, a)$ is countable, we put $B = A$;
Case 2: if there exists some a for which $(-\infty, a)$ is uncountable, let b be the smallest such a (which exists as A is well-ordered) and put $B = (-\infty, b)$.
It is immediate that (in both cases)
(1) B is well-ordered (it is a subset of A);
(2) for all x in B, $(-\infty, x)$ is countable;
(3) B is uncountable;
(4) for each x in B, there is a unique x' in B such that $x' < x$ and $(x, x') = \emptyset$.
Note that (4) holds because given x in B, (2) and (3) imply that there is some t in B with $x < t$, and the well-ordering implies that the set of all such t has a smallest member x'. The key properties of B are expressed in the next result.

Theorem 5.1. *The ordered space $(B, <)$ has the properties*
(a) for each x in B, the order $<$ on $(-\infty, x)$ can be represented by a real-valued function;
(b) the order $<$ cannot be represented by a real-valued function on B.

Proof: It is well-known (and easy to prove) that any countable set can be mapped in an order-preserving way into the set of rational numbers; thus (a) follows. On the other hand, $<$ cannot be represented on B by a real-valued function u, for if it could, then the intervals $(u(x), u(x'))$ as x varies over B would constitute an uncountable set of pairwise disjoint subintervals of $\mathbf{R}$ and we know that this cannot happen. Note that (b) is also a consequence of Theorem 4.1 as, by (3) and (4), B has uncountably many jumps.

There are results which suggest that, under fairly weak conditions, any preference relation which cannot be represented by a utility function must look (roughly speaking) rather like either the lexicographic order or the long-line ([17]). This seems to me to be an area worthy of much more study; there is a need to understand and illustrate the reasons why a preference relation cannot be represented by a utility function.

6. The existence of continuous utility functions

It is a rather surprising fact that the continuity of utility functions with respect to the order topology is not an issue at all. We have

Theorem 6.1. *If $\preceq$ can be represented by a utility function on X, then it can also be represented by a utility function that is continuous with respect to the order topology on X.*

Clearly, if $\preceq$ can be represented by a continuous utility function u on X, then $h(u(x))$ is also a continuous utility function for every continuous map $h : u(X) \to \mathbf{R}$. It is natural to identify (that is, not distinguish between) the functions u and hu when h is a homeomorphism, and with this we have the following uniqueness result.

Theorem 6.2. *Any two continuous utility functions representing $\preceq$ differ by composition with a homeomorphism.*

The original proof of Theorem 6.1 was given by the economist Debreu (1952) who, for this purpose, proved

Theorem 6.3: the Gap Theorem. *Let E be a subset of $\mathbf{R}$. Then there is a strictly increasing map $\phi : E \to \mathbf{R}$ such that the complement of $\phi(E)$ has no bounded components of the form $[a, b)$ or $(a, b]$.*

The significance of the Gap Theorem is that if we have a utility function u on X, and if we take $E = u(X)$, then we find that ϕu is a continuous utility function. The original 'proof' of the Gap Theorem was simply to 'collapse' all of the offending intervals $[a, b)$ and $(a, b]$ to a single point: however, it was soon realized that this argument is not valid because in some cases (in which E has measure zero) this would also collapse E to a single point. A valid argument seems to require (in one form or another) a process which *at the same time* 'expands' E (possibly from zero measure to positive measure) and collapses the intervals in its complement. There are now a variety of proofs of the Gap Theorem available (see [3], [5], [7], [9], [13], [16] and [20]), and the next result [5] (based on the fundamental paper [21] by the economist Wold in 1943) contains, as a Corollary, the Gap Theorem and several other important results in this area.

Theorem 6.4. *Let $\sim$ be an equivalence relation on the closed interval $[0, 1]$ with the property that each equivalence class is a closed interval. Then there is an increasing continuous function $u : [0, 1] \to [0, 1]$ such that $u(x) = u(y)$ if and only if $x \sim y$.*

To relate the Gap Theorem to a more concrete example, consider for the moment a strictly increasing map f of $[0, 2]$ to $[0, 1] \cup (2, 3]$. The map is not continuous (because the complement of its range has a component that is a half-open half-closed interval) and, equally importantly, the set $[0, 1]$ is open in the subspace topology of $f(E)$ but *not* in the intrinsic order topology (namely, the order topology derived from the induced order on $f(E)$). More generally, given a subset K of a linearly ordered space X, K carries both the subspace topology and its own intrinsic order topology (found by first restricting the order to K and then creating the corresponding order topology), and these two topologies need not be the same. The 'gap condition' on the range $\phi(E)$ in the Gap Theorem is precisely the condition that these two topologies coincide.

These ideas lead to a study of general questions about continuity of increasing functions between linearly ordered spaces, and we can establish the following general result (see [4]).

Theorem 6.5. *Let X and Y be linearly ordered spaces, and suppose that the subspace topology and the intrinsic order topology on a subset E of X are the same. Then for every strictly increasing map $f : E \to Y$, the map $f^{-1} : f(E) \to E$ is continuous (with respect to the subspace topologies).*

Observe first that the result generalizes the well-known elementary result that if $f : [a, b] \to \mathbf{R}$ is strictly increasing, then $f^{-1} : f([a, b]) \to [a, b]$ is continuous (most elementary texts assume, unnecessarily, that f is continuous). Indeed, this follows from Theorem 6.5 because on any compact interval the subspace and intrinsic order topologies coincide.

Next, if we consider an increasing map $f : E \to f(E)$ which has the property that for both E and $f(E)$ the subspace and intrinsic order topologies coincide, then, by Theorem 6.5, f is a homeomorphism from E to $f(E)$. If this is taking place in the context of the real line, the statement about the two topologies can be replaced by a statement about half-open half-closed gaps and this leads, ultimately, to the uniqueness expressed in Theorem 6.2.

7. Utility functions with values in a linearly ordered group

A major unresolved question is *what can be said if a given preference relation cannot be represented by a utility function?* From a purely mathematical point of view, the insistence that utility functions be real-valued is bound to lead to difficulties for, after all, if X is 'very' large, then the range of any utility function on X must necessarily be correspondingly large to cope with this, and in many case it will be significantly larger than $\mathbf{R}$. Indeed, it can be

argued that to reject utility functions whose range is 'larger' than $\mathbf{R}$ is simply refusing to tackle the real issue. It is natural, then, to try to develop a theory of utility functions whose range can be any suitably large ordered set. Without further restrictions, however, this is an empty problem (for each ordered set X can be represented by the identity map into itself, or by the quotient map onto $X/\sim$). We need, then, to find a class of ordered sets with some additional structure, and then allow a utility function to map into one of these sets.

One possibility is to seek utility functions with values in a suitable *linearly ordered group* (that is a group which is ordered in a way that is compatible with the group operation) and to help the theory along, there is a large theory of linearly ordered groups available, [15]. Moreover, the theory of linearly ordered groups contains much on lexicographically ordered products of groups, so there does seem to be a strong link here with the earlier discussion. It is perhaps worth noting that with its order topology, a linearly ordered group becomes a topological group (that is, the operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous).

In fact, it is known that any preference relation $\preceq$ on any space X *can* be represented by a utility function with values in an abelian linearly ordered group. To see this, we take the group of integer-valued functions on X which are zero except at a finite set of points of X. The group operation is addition in the usual way, and we write $f < g$ if $f(x) < g(x)$, where x is the least preferable point of X at which f and g disagree

We have not really solved our problem, however, for as a linearly ordered group is an ordered space, it supports its own order topology and knowing this, we must surely seek the existence of a *continuous* utility function. In this respect, there is an amusing, and tantalizing, observation to be made. The very example that is universally quoted as a preference relation which cannot be represented by a utility function, namely the lexicographic order on the first quadrant, is itself a linearly ordered group with the group operation

$$(x, y) \oplus (u, v) = (xu, yv).$$

This means, of course, that the lexicographic preference relation *can be represented by a continuous utility function with values in an abelian linearly ordered group*, namely the identity function mapping Ω onto itself!

We end with an avenue for further study. The Gap Theorem has been proved for maps from **R** to **R**, and, in the light of the remarks just made, we now ask for which linearly ordered groups is there a corresponding 'Gap Theorem' available? If we could show that **R** is the only linearly ordered group for which such a result is true, then this would provide a *mathematical* justification for restricting our attention to real-valued utility functions. If, on the other hand, there were other groups for which such a result existed, it might lead to a theory of more general utility functions which would, on mathematical grounds at least, have an equal claim to our attention.

References

[1] K. J. Arrow and F. Hahn, General Competitive Analysis. Oliver and Boyd: 1988.

[2] K. J. Arrow and M. D. Intriligator, Handbook of Mathematical Economics, Volume II. North Holland: 1982.

[3] A. F. Beardon, *Debreu's Gap Theorem*, Economic Theory **2** (1992a), 150-152.

[4] A. F. Beardon, *Utility theory and continuous monotonic functions*, Economic Theory, to appear.

[5] A. F. Beardon and G. B. Mehta, *The utility theory of Wold, Debreu and Arrow-Hahn*, Econometrica, to appear.

[6] G. Birkhoff, Lattice Theory. Amer. Math. Soc. Pub. 25: New York, 1948.

[7] R. Bowen, *A new proof of a theorem in utility theory*, International Economic Review **9** (1968), 374.

[8] J. W. S. Cassels, Economics for Mathematicians. London Math. Soc. Lecture Notes, No. 62. Cambridge University Press: 1981.

[9] G. Debreu, *Representation of a preference ordering by a numerical function*, *in* Decision Processes, ed. by R. Thrall, C. Coombs and R. Davis, 159-165. Wiley: 1954.

[10] G. Debreu, Mathematical Economics, Chapters 6 and 12. Cambridge University Press: 1983.

[11] S. Eilenberg, *Ordered topological spaces*, Amer. J. Math. **63** (1941), 39-45.

[12] I. Fleischer, *Numerical representations of utility*, Journ. S.I.A.M **9** (1961), 48-50.

[13] J. Jaffray, *Existence of a continuous utility function: an elementary proof*, Econometrica **43** (1975), 81-83.

[14] E. Klein and A. C. Thompson, Theory of Correspondences. Wiley Interscience: 1984.

[15] A. I. Kokorin and V. M. Kopytov, Fully Ordered Groups. Wiley: 1974.

[16] G. B. Mehta, *Recent developments in utility theory*, Indian Economic Journal **30** (1983), 103-124.

[17] P. K. Monteiro, *Some results on the existence of utility functions on path connected spaces*, J. Mathematical Economics **16** (1987), 147-156.

[18] L. Nachbin, Topology and Order. Van Nostrand: 1965.

[19] T. Rader, *The existence of a utility function to represent preferences*, Review of Economic Studies **30** (1963), 229-232.

[20] M. Richter, *Continuous and semi-continuous utility*, International Economic Review **21** (1980), 293-299.

[21] H. Wold, *A synthesis of pure demand analysis, I, II and III*, Scandinavisk Aktuarietidskrift **26** (1943-44), 85-118, 220-263, 69-120.

A. F. Beardon,
Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge,
16 Mill Lane,
Cambridge CB2 1SB,
England.