Lastly the index bundle and Bismut's index theorem are considered. The *index bundle* is defined for a family of Dirac operators by considering a fibre bundle $\pi : M \to B$ with fibres denoted by M/B. For every $z \in B$, $M_z = \pi^{-1}(z)$ is the fibre over z and $D = \{D^z | z \in B\}$ is a family of Dirac operators on M/B. If $\ker(D^z)$ has the same rank for each z the vector spaces $\ker(D^z)$ combine to form a vector bundle over B called the index bundle, $\mathrm{ind}(D)$. The construction can be generalized to the case where the rank of $\ker(D^z)$ depends on z. Bismut's index theorem then relates the character of a superconnection for the family D to an integral over M/B. It is of interest to physicists as it has proved to be useful in string theory and the theory of moduli spaces of Yang-Mills fields.

In addition there are general chapters on equivariant differential forms and the exponential map, relating the $\hat{A}$ genus to the Jacobian of the exponential map of the Lie algebra of $SO(n)$, as well as a section on zeta functions.

In summary I found the book stimulating and rewarding, as it brought me a little more up to date in a subject which I know a little about but am not an expert in, but a thorough reading and understanding would require a larger investment of time than I can presently afford.

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Book Review

REPRESENTATION THEORY. A FIRST COURSE

William Fulton and Joe Harris

Springer–Verlag, New York, Berlin, Heidelberg, 1991. xv+551 pp.
ISBN 0-387-97495-4.

Reviewed by Rod Gow

Let me say right at the beginning that I think the authors Fulton and Harris have produced an excellent book, a book which displays a novel approach to its subject matter and is genuinely informative. Too often one feels that a textbook is largely a recitation of known techniques and ideas, with little evidence that the author has tried to find worthwhile examples or new approaches to difficult problems. However, this book presents a substantial amount of unfamiliar material in a way that is pleasing to a mathematician who has a reasonable knowledge of modern concepts of algebra.

The main subject matter of the book under review is the description of the irreducible complex representations of the simple Lie algebras over **C**. While we have a description of these representations in terms of highest weight modules, due to É. Cartan and Weyl, the emphasis in the book is directed towards explicit realization of the representations wherever possible, using the methods of multilinear algebra, symmetric polynomial theory and invariant theory. More detail is expended on the classical Lie algebras, which fall into four infinite families, since these are more accessible as algebras of vector space endomorphisms and their representations may be studied rather more explicitly than those of exceptional algebras such as F_4, E_6, E_7 and E_8. A worthwhile feature of the authors' approach is the way they are able to point out interesting geometric aspects of the representations they construct. Thus, under the guise of geometric plethysm, arising in the context of the representations of $sl_2(\mathbf{C})$ and $sl_3(\mathbf{C})$, there is a discussion of the Veronese embedding of a projective space in a larger projective space and of the associated Veronese surfaces. Grassmannian varieties also occur, with a description of their Plücker embeddings into projective spaces, when the decomposition of symmetric powers of exterior powers is examined.

The authors organize the material of the book into 26 lectures, rather than chapters. There are numerous exercises throughout the book, some of which are distinctly demanding of the reader. There are, however, at the end of the book, 20 pages of hints, answers and references relating to the exercises. There are 63 pages of appendices concerning such topics as symmetric polynomials, multilinear algebra, properties of Lie algebras and classical invariant theory. There is also a bibliography of six pages. The book begins with familiar material on the complex representations of finite groups. However, more emphasis is given than usual to the problem of decomposing certain tensor products for particular groups. Indeed, already on p.31, the authors show that the exterior powers of the natural module of degree $d-1$ for the symmetric group S_d of degree d are all irreducible. While this is a known result, a head-on proof of the kind given is not usually found in the literature. Lecture 4 is devoted to the construction of the irreducible characters of S_d. This is of course a vast area of study, where numerous contributions have been made. I enjoyed the authors' presentation, which includes many interesting facts and different points of view. In Appendix A at the end of the book, there are proofs of a number of results concerning symmetric polynomials which are needed to develop the finer aspects of the character theory of S_d. In particular, Schur functions are introduced and various determinantal formulae for their evaluation are proved. The Littlewood–Richardson rule for multiplying Schur functions is described but not proved. However, a special case, known as Pieri's rule, is proved and this is often sufficient for many purposes. It is clear that much of classical determinant theory, such as one sees in Thomas Muir's *Treatise on the theory of determinants*, finds its natural home in this context. Lecture 6 presents the theory of Schur and Weyl for decomposing the n–fold tensor power $V^{\otimes n}$ of a finite dimensional complex vector space V into irreducible $GL(V)$–submodules, where $GL(V)$ is the group of all automorphisms of V. The irreducible submodules are picked out as the images of the Young symmetrizers, which are rational multiples of certain idempotents in the rational group algebra of S_n. Examples are given to show how the Littlewood–Richardson rule is used for working out, among other things, the decomposition of certain tensor products.

Part II of the book is devoted to introductory material on Lie algebras and Lie groups. While a Lie algebra is an easily defined algebraic object, a Lie group seems much more complicated, with its attendant topology and geometry. The authors make the point that the structure of the Lie algebra of a Lie group provides crucial information on the structure of the group. Moreover, the finite dimensional irreducible representations of the group may be studied via the irreducible representations of the algebra. Standard examples of real and complex linear Lie groups are given, together with less familiar examples of complex tori, related to elliptic curves and abelian varieties. It is shown how Lie algebras arise from Lie groups by taking the differential of the adjoint representation of the Lie group on the tangent space of the identity and then how to pass back to (subgroups of) the Lie group via the exponential map. A number of the basic theorems on complex Lie algebras are proved, such as those of Lie, Engel and Cartan. The Killing form is introduced somewhat later in the book. Then the irreducible representations of the most accessible of the simple Lie algebras over $\mathbf{C}$, $sl_2(\mathbf{C})$, are investigated. This material, while straightforward, is vital for understanding the structure of arbitrary simple Lie algebras and also plays a role in the representation theory, via the principal 3–dimensional Lie algebras that occur as subalgebras of the simple algebras. In order to describe the irreducible representations of $sl_3(\mathbf{C})$, it is necessary to develop certain ideas, such as highest weight vector and weight lattice, that play the dominant role in the representation theory of all simple Lie algebras.

Part III of the book develops some more theory relating to simple Lie algebras, special cases of which were encountered when investigating $sl_2(\mathbf{C})$ and $sl_3(\mathbf{C})$. The irreducible representations of the special linear, symplectic and orthogonal Lie algebras are constructed and investigated from a variety of points of view, which I found most instructive and helpful. In order to understand the missing representations of the orthogonal Lie algebras, which cannot be constructed from tensor operations on the natural module, the authors introduce the Clifford algebra and spin groups. This material again is handled in a very clear conceptual manner. There is also a brief discussion of the triality automorphism of order 3 in the Lie algebra $so_8(\mathbf{C})$. Part IV of the book contains a variety of material, which we can scarcely do justice to. Among more familiar topics are Weyl's character formula, the weight multiplicity formulae of Freudenthal and Kostant and Cartan's classification of real simple Lie algebras. Less familiar topics discussed include the connection between g_2 and skew–symmetric trilinear forms defined on a seven–dimensional vector space and algebraic constructions, due to Freudenthal, of the exceptional Lie algebras, where, again, trilinear forms play a basic role. The relationship between the Clifford algebra based on an eight–dimensional vector space, octonions (Cayley numbers), g_2 and the triality automorphism is also briefly explained (g_2 is, as far as I know, the fixed algebra of the triality automorphism).

There are a number of good introductory books on Lie algebras and their representations, such as those Jacobson and Humphreys. Bourbaki, in his three volumes on Lie theory, provides a large amount of material as well, much unavailable elsewhere, but the reader is often asked to find proofs from minimal hints. I think that this book, with its concentration on examples to illustrate and interpret the theory, is a most useful addition to the literature in this area. Its style will appeal to pure mathematicians but people, such as physicists, who occasionally have recourse to Lie theory, should also find something worthwhile for them when studying this text. I did not notice very many obvious typographical errors in the book. The authors seem to have prepared the typescript themselves by computer and this may account for the good quality of the work. The

authors cannot decide on the spelling of octonion, which appears variously as octonion and octonian, sometimes on the same page. The name of the Italian mathematician Trudi is given as Trudy and Gordan, of Clebsch–Gordan fame, has his name consistently misspelt as Gordon. The word principle appears for principal on p.422 and there is a $\wedge$ sign missing from formula B7 on p.474. However, the overall impression created in this reviewer was that of an ambitious text, skilfully worked and interspersed with novel observations, which any library or researcher, experienced or novice, might purchase without regret.

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