Book Review

HEAT KERNELS AND DIRAC OPERATORS

Grundlehren der mathematischen Wissenschaften

N. Berline, E. Getzler and M.Vergne

Springer-Verlag, 1992
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Reviewed by Brian Dolan

This book provides a modern unified approach to index theorems for elliptic operators on compact manifolds. The book grew out of a seminar in 1985 at MIT and would be useful both for researchers with some prior knowledge of differential geometry, wishing to deepen their understanding, as well as for workers in the field. It is mathematical in its approach, as befits its subject, and assumes familiarity with such topics as fibre bundles, connections and cohomology theory. Despite devoting the first fifty or so pages to general background on differential geometry I don't think it would be a good place for someone with no previous knowledge of the subject to learn the fundamentals. The aim seems to be to set up the notation rather than explain the concepts, and the notation rapidly becomes quite involved. The book is carefully laid out in logical sequence and a little careful study is well rewarded. Typographical errors are rare, but do exist. The index is rather short, but seems adequate. There are one hundred references to some of the most important publications in the subject and the authors freely admit that this is by no means exhaustive. Only six of the references predate 1960 which gives some indication of of the historical development of the subject and it is amusing to note that, despite the title, there is no mention of Dirac's 1928 paper! There is an extensive list of notations at the back of the book which I

found extremely useful, in fact indispensable, in trying to find my way through the symbols. An indication of the style of the book is given by the fact that the index is four and a half pages long while the list of notations is three and a half pages long.

After the first chapter a general framework is developed, in terms of generalized Dirac operators on vector bundles with a $\mathbf{Z}_2$ grading, giving rise to the concept of a *supertrace* on the space of fibre endomorphisms and a *superconnection* on the vector bundle which is a first order differential operator odd under the $\mathbf{Z}_2$ grading. The definition of a Dirac operator that the authors adopt is general enough to encompass all the usual first order operators. A key ingredient of their construction is the one to one mapping between the space of exterior forms on a differentiable manifold M and the Clifford algebra for a vector space with a metric, when the vector space is viewed as a fibre of the tangent bundle of M, which they refer to as the *quantization* map.

The index theorem of Atiyah and Singer is proven, using the heat kernel approach, and its application to the four classic complexes is exhibited, giving proofs of the Gauss-Bonnet theorem, the Hirzebruch signature theorem, the index theorem for the Dirac operator and the Riemann-Roch-Hirzebruch theorem. The authors then go on to treat the equivariant index theorem, which generalizes the Atiyah-Singer index theorem to the case where there is a group action on the manifold M which is compatible with (i.e. commutes with) the generalized Dirac operator. Thus the kernel of the Dirac operator forms a representation space for the group. The equivariant index is then a generalization of the character of a group element in a given representation, its supertrace, and the equivariant index theorem relates this to an integral over the fixed point set of the group action, which is a subset of M for non-trivial group actions. Along the way the authors dispose of the Atiyah-Bott fixed point formula, where the fixed points consist of isolated, non-degenerate points. They then go on to state and prove a version of the equivariant index theorem which holds when the group element is near the identity, which they term the Kirillov formula, by analogy with Kirillov's formulas for the characters of Lie groups.

Lastly the index bundle and Bismut's index theorem are considered. The *index bundle* is defined for a family of Dirac operators by considering a fibre bundle $\pi : M \to B$ with fibres denoted by M/B. For every $z \in B$, $M_z = \pi^{-1}(z)$ is the fibre over z and $D = \{D^z | z \in B\}$ is a family of Dirac operators on M/B. If $\ker(D^z)$ has the same rank for each z the vector spaces $\ker(D^z)$ combine to form a vector bundle over B called the index bundle, ind(D). The construction can be generalized to the case where the rank of $\ker(D^z)$ depends on z. Bismut's index theorem then relates the character of a superconnection for the family D to an integral over M/B. It is of interest to physicists as it has proved to be useful in string theory and the theory of moduli spaces of Yang-Mills fields.

In addition there are general chapters on equivariant differential forms and the exponential map, relating the $\hat{A}$ genus to the Jacobian of the exponential map of the Lie algebra of $SO(n)$, as well as a section on zeta functions.

In summary I found the book stimulating and rewarding, as it brought me a little more up to date in a subject which I know a little about but am not an expert in, but a thorough reading and understanding would require a larger investment of time than I can presently afford.

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Book Review

REPRESENTATION THEORY. A FIRST COURSE

William Fulton and Joe Harris

Springer–Verlag, New York, Berlin, Heidelberg, 1991. xv+551 pp.
ISBN 0-387-97495-4.

Reviewed by Rod Gow

Let me say right at the beginning that I think the authors Fulton and Harris have produced an excellent book, a book which displays a novel approach to its subject matter and is genuinely informative. Too often one feels that a textbook is largely a recitation of known techniques and ideas, with little evidence that the author has tried to find worthwhile examples or new approaches to difficult problems. However, this book presents a substantial amount of unfamiliar material in a way that is pleasing to a mathematician who has a reasonable knowledge of modern concepts of algebra.

The main subject matter of the book under review is the description of the irreducible complex representations of the simple Lie algebras over **C**. While we have a description of these representations in terms of highest weight modules, due to É. Cartan and Weyl, the emphasis in the book is directed towards explicit realization of the representations wherever possible, using the methods of multilinear algebra, symmetric polynomial theory and invariant theory. More detail is expended on the classical Lie algebras, which fall into four infinite families, since these are more accessible as algebras of vector space endomorphisms and their representations may be studied rather more explicitly than those of exceptional algebras such as F_4, E_6, E_7 and E_8. A worthwhile feature of the authors' approach is the way they are able to point out interesting geometric aspects of the representations