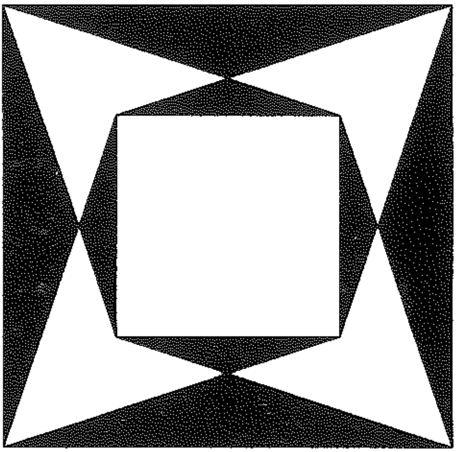
IRISH MATHEMATICAL SOCIETY



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IRISH MATHEMATICAL SOCIETY BULLETIN

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, in March and December. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files if possible, in order to ensure speedier processing.

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1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society and the Irish Mathematics Teachers Association.

2. The current subscription fees are given below.

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6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription fee to

   The Treasurer, I.M.S.
   Department of Mathematics
   University College
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Minutes of Meetings of the Irish Mathematical Society

Ordinary Meeting

16 April 1992

An ordinary meeting of the Irish Mathematical Society was held at 12.15 pm on April 16th 1992 at DIAS.

The President (R. Timoney) took the chair and there were 7 other members present.

1. Minutes

The Minutes of the meeting of 20th December 1991 were approved and signed.

2. Matters Arising

No matters arising.

3. Bulletin

M. Ó Searcóid reported on a really wonderful improvement in the production of the Bulletin, achieved with the help of Rex Dark of UCG (acting editor). The March '91 issue was with the printers, the December '91 issue was nearly done and would be ready by the time the March '91 issue came back from the printers. MÓS had also designed a new format file (in plain TeX) which had many useful features and incorporated an automatic system for keeping track of cross-references. MÓS would make his format file publicly available and encourage authors to use it.

The meeting warmly thanked Micheál Ó Searcóid for his excellent work and also expressed its thanks to Rex Dark.

4. European Mathematical Society

Fees from individual members of the IMS wanting to join the EMS had been collected and forwarded to the EMS.

Several reports were required for the European Congress of Mathematicians in July (nearly every 'Table Ronde' had requested some sort of information). This was being collected or had been collected.

A request to nominate Irish mathematicians willing to serve as referees for the evaluation of network applications to the Human Capital and Mobility scheme was discussed. A letter asking the IMS to suggest suitable names had been received from the EMS and that letter also asked for an Irish person willing to serve on the 'Mathematics panel' for networks. It was agreed that the Committee would discuss the matter further before deciding on what to do.

The advantages to be offered under the HCM scheme were pointed out to members.

5. Treasurer's business

The proposal for a change in the financial year to correspond with the calendar year was approved by the meeting following a motion proposed by D. Tipple and seconded by M. Ó Searcóid.

The Treasurer, D. Tipple then produced a report on the interim period October – December 1992 between the end of the financial year under the old system and the beginning of the financial year under the new system. This was accepted.

6. EUROMATH report

R. Timoney gave a brief report on the recent status of the EUROMATH project with the aid of a few slides, in his capacity as the chairman of the National Coordinating Committee for EUROMATH in Ireland. R. Timoney is also a member of the Project Committee of EUROMATH, which is charged with bringing the project work to a successful conclusion.

The overall objective of the EUROMATH Project is to promote the use of modern information technology among mathematicians in Europe.

The project is run by the European Mathematical Trust, an organization founded by the mathematical societies of 18 'European' countries (actually EC + EFTA countries). There is participation by Eastern European countries also, as observers. The project is funded under the SCIENCE program.

For what purposes do mathematicians use computers? Answers:

1. Document preparation
2. Symbolic and numeric computation
3. Database search
4. Electronic mail etc.
5. ...

Looking at databases for as moment (RT is in charge of overseeing the work being done by EUROMATH in this area), one might come up with the following ideal scenario:

1. access to online versions of Math Reviews and Zentralblatt would be cheap and simple;
2. formulae in screen views of an abstract from an online database would be easy to read;
3. references from a database search could be directly inserted in the list of references of a paper;
4. remote, local and personal databases would be accessible using the same query language.

The project hopes to realize all of these items (except that it depends partly on external bodies to completely REALIZE the objective of viewing Zentralblatt formulae on screen, for example).

On the question of document preparation, the project has identified a trend towards the use of SGML (Standard Generalized Markup Language) by publishers, database providers and others. The Project aims to cater for this trend (as well as LaTeX typesetting) by producing a WYSIWYG editor which would combine the usual advantages of WYSIWYG systems with the power and flexibility of Markup-based systems like LaTeX. Here is a short comparison of the usual WYSIWYG and Markup systems:

WYSIWYG:	**Markup:**
* Complete freedom in layout	* Coherent and well designed documents
* Requires experience of document design	* Can lead to excessive uniformity
* Direct manipulation on screen	* Structure present in
* WYSIAYG on screen	* source
	* Edit-typeset-preview
	* loop

(WYSIAYG = what you see is all you get.)

The whole EUROMATH system is being designed around this editor (using the editor as a coherent user interface in fact) and the schedule is for completion before mid 1993.

In the meantime Mathematics Institutes are being invited to participate in the project by subscribing to EUROMATH. Subscribing now will entitle institutes to receive an early version of the EUROMATH editor right away (for either SUN4, SUN3, DEC 3100/5000 or IBM RS600 workstations) plus getting the production version when it is ready in 1993. Also institutes will benefit from a special rate for the online version of the Zentralblatt. The cost of EUROMATH subscriptions for the period until the end of 1993 has been fixed at 1500 ECU for institutes who subscribe before the end of 1992 and 2000 ECU for those subscribing later.

Some of those present felt this was a lot of money.

7. Any Other Business

It was noted that two reports had appeared in the UK which might be of interest to members. One was 'The future for Honours Degree Courses in Mathematics and Statistics' (Feb 1992) by a group working under the auspices of the LMS and with the support of the Royal Statistical Society and the Institute of Mathematics and its Applications. It dealt with 3- versus 4-year degrees in the English context (excluding computing).

The second was the so-called Kingman report, officially entitled 'Mathematics strategy for the future', published by SERC and dealing more with support for research.

Richard Timoney
Trinity College,
Dublin.

GROUPS 1993 GALWAY / ST ANDREWS

An international conference on groups will be held in Galway from 1st to 14th August 1993. It will be the next in the sequence Groups–St. Andrews 1981, 1985, 1989. The speakers will include J. L. Alperin (University of Chicago), M. Broué (École Normale Supérieure, Paris), P. H. Kropholler (QMW, London), A. Lubotzky (Hebrew University, Jerusalem) and E. I. Zel'manov (University of Wisconsin at Madison). A GAP workshop will be led by J. Neubüser and M. Schönert (RWTH Aachen). Further information may be obtained:

from C. M. Campbell or E. F. Robertson, Mathematical Institute, University of St Andrews, St Andrews KY16 9SS, Fife, Scotland (e-mail: `groups93@cs.st-andrews.ac.uk`);

or from J. J. Ward, T. C. Hurley, or S. J. Tobin (Honorary President) University College, Galway, Ireland
(e-mail: `matward@bodkin.ucg.ie`).

GROUPS IN GALWAY 1993

The annual Groups in Galway meeting will take place in University College Galway on Friday 14th and Saturday 15th May 1993. Further information will be available from Dr John McDermott, Department of Mathematics, University College, Galway, or from `matnewell@bodkin.ucg.ie`

THIS BODE'S ILL FOR BOOLE

Dear Sir,

The 'Numerical Recipes' books of Press et al. have gained a wide following and look set to become standard classics. The same basic book appears in FORTRAN, Pascal, and C flavours, with accompanying diskettes of canned software. The no–nonsense approach is designed to appeal to practical people, and one finds the book on the shelves of researchers and teachers in every scientific and technical area.

It follows that the opinions of Press et al. are likely to become gospel, and the nomenclature used by them is likely to become standard.

Now in fact, their opinions are just their opinions and although some of the recipes are simply awful, what prompts us to action is a curiosity of their nomenclature.

They introduce integration rules by presenting the Trapezoidal Rule, Simpson's Rule, and Bode's Rule, and then say:

> "At this point the formulas stop being named after famous personages, so we will not go any further".

So who is this famous personage, Bode?

Given the fact that Bode's rule is identical with Boole's rule, we are led to conjecture that Bode and his rule came into being through the close juxtaposition on some blackboard of an 'o' and an 'l'.

Boole has been robbed by a phantom!

In view of Boole's association with this country, it would seem appropriate for us to nail this particular piece of thievery before it goes too far, if indeed it has not already done so. Frankly, we are inclined to doubt that the truth will ever catch up, but we might as well try.

J. F. Feinstein,
A. G. O'Farrell,
St Patrick's College,
Maynooth,
Co. Kildare.

GROUPS IN GALWAY 92

Rex Dark

The annual Groups in Galway meeting was held on 15th and 16th May 1992, with the help of sponsorship from the Irish Mathematical Society, the Royal Irish Academy, and University College, Galway. There were about twenty participants.

The first lecture was by A. Christofides (UCG) on "Galois groups and Riemann surfaces". After tea, R. Sheehy (UCC) spoke on "Frobenius' conjecture", and the last lecture on Friday was by H. Smith (Bucknell and Cardiff) on "Some remarks on maximal subgroups of infinite groups".

The Saturday morning session was begun by K. Hutchinson (UCD) who spoke on "Galois group actions on classgroups", followed by E. Robertson (St Andrews) on "Semigroup presentations". After lunch, the meeting closed with a lecture by B. Hartley (Manchester) on "Simple locally finite groups".

Rex Dark
University College
Galway

THE 1992 IMS SEPTEMBER MEETING

Michael Brennan and Brendan McCann

The 1992 September meeting took place on Thursday 3rd and Friday 4th September at Waterford Regional Technical College, and was organized by the mathematics staff at WRTC. There were over 40 participants at the meeting, and it was very encouraging to note that many of the lectures were attended by computer science, engineering and physics staff at WRTC.

After the opening address by the principal of WRTC, Mr Ray Griffin, the first speaker was Professor Darrel Ince (Department of Computing, Open University). In his talk, *Discrete mathematics and the formal development of programs,* he outlined some of the advances and retreats of formal (that is, mathematical) methods of program specification and verification. The second talk of the opening session was given by Dr John McDermott (UCG). It was entitled *Colouring problems,* and in it he described some of the problems involved in determining the chromatic number (that is, the minimum number of colours needed to distinguish the vertices) of a directed graph.

After lunch on the first day, Professor David Armitage (QUB) spoke on *Harmonic functions: background and recent results.* In his talk he reviewed the development of the theory of harmonic functions, and included an account of some of the new results of the past few years. Then Mr Éamonn de Leastar (WRTC) presented a talk on *Numerical data types in C and C^{++},* in which he discussed the differences between the two languages, using as an example a simple program specification.

There followed a coffee break, after which Professor Gilbert Strang (Massachusetts Institute of Technology) discussed the case of *Wavelet transforms vs. Fourier transforms.* After presenting

examples, an application to high resolution television, and a review of the mathematical theory, he concluded that the Fast Fourier Transform will not, in most cases, be superseded by the Fast Wavelet Transform. The final talk of the first day, *Numerical solution of convection-diffusion problems,* was given by Dr Martin Stynes (UCC). He talked about some of the very difficult numerical problems encountered when trying to approximate the solution to the second order differential equations which model physical situations with "dominant" convection and secondary diffusion.

The second day of the meeting began with *Higher order symmetry of graphs,* a talk given by Professor Ronald Brown (University of North Wales, Bangor). He described a category theoretical approach to the idea of symmetry, involving the category of digraphs, appropriate morphisms, and the general notion of a *topos* in place of a set. Then Professor Joaquín Gutiérrez (Università Politecnica di Madrid) talked on *Polynomials and series in Banach spaces,* and dealt with those series in Banach spaces whose convergence is preserved under certain polynomial mappings. After coffee, Professor John Lewis (DIAS) spoke on *Thermodynamic aspects of probability theory.* He discussed Boltzmann's equation: $S = k \log W$ (where S is the entropy of an equilibrium state, and W is the number of microscopic states corresponding to that equilibrium state), and the corresponding probability theory namely Large Deviation Theory.

The afternoon session began with Professor Strang's second talk *New ideas on teaching calculus, linear algebra and applied mathematics,* in which he gave a flavour of his own teaching style by going through a number of easy-to-follow examples. Then Dr Patrick Fitzpatrick (UCC) talked about *A theoretical basis for Padé approximation,* where the problem is to approximate a polynomial of given degree by the quotient of 2 polynomials of lesser degree. He discussed applications to coding theory, and showed how Gröbner bases help to resolve some of the problems.

After coffee Mr Christopher Boyd (UCD) gave a talk on *Preduals and linearization of holomorphic mappings,* which dealt with dual-nuclear spaces of holomorphic functions. The meeting then closed with an address by the President of the IMS, Dr Richard Timoney.

The organizers would like to thank all the speakers and participants at the meeting, especially those who chaired the sessions. In addition they would like to thank Waterford Regional Technical College and the City of Waterford Vocational Education Committee for their generous funding, and the Waterford branch of the Bank of Ireland who provided a contribution towards the running costs of the meeting.

Michael Brennan and Brendan McCann
Regional Technical College
Waterford

THE CONTINUITY OF THE SEMI-FREDHOLM INDEX

Mícheál Ó Searcóid

Introduction

It is well-known and easy to prove that the index function is continuous on the set of Fredholm operators on a Banach space. It is also true that the index function is continuous on the larger set of semi-Fredholm operators. The proof presents no difficulty in the case where the semi-Fredholm operators are simply those operators which are left or right invertible modulo the compact operators, as happens in the case where the Banach space is a Hilbert space. The usual proofs in the more general context [4, p.60], [1, pp.62-63] use the notion of gap between between subspaces and require more preliminary work than one might have suspected necessary. In this note we show how to avoid such unsatisfactory excursions by giving a natural operator-theoretic proof of this basic result. The nature of the proof makes it convenient to consider the slightly more general case in which the operators act between two possibly different spaces.

1. Preliminaries

If X and Y are Banach spaces, then $\mathcal{B}(X,Y)$ will denote the set of bounded linear operators from X to Y, and $\mathcal{F}(X,Y)$ will denote the set of finite rank operators in $\mathcal{B}(X,Y)$. When $X = Y$ we shall write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,Y)$ and $\mathcal{F}(X)$ instead of $\mathcal{F}(X,Y)$; all similar notation will be abbreviated in the same way. Identity operators on spaces will be denoted by I; the space in question will always be obvious from the context. For each $T \in \mathcal{B}(X,Y)$, the nullity $\mathrm{nul}(T)$, and the defect $\mathrm{def}(T)$ of T are defined to be the dimension of the nullspace of T and the codimension in Y of the closure of the range of T respectively. Provided not both these quantities are infinite, we define the index of T by

$$\text{ind}(T) = \text{nul}(T) - \text{def}(T).$$

We shall be concerned with two sets of closed range operators, the sets of upper and lower semi-Fredholm operators, defined respectively to be

$\Phi_+(X,Y) = \{T \in \mathcal{B}(X,Y) : T(X) \text{ closed in } Y \text{ and } \text{nul}(T) < \infty\}$

and

$\Phi_-(X,Y) = \{T \in \mathcal{B}(X,Y) : T(X) \text{ closed in } Y \text{ and } \text{def}(T) < \infty\}$.

Their intersection, denoted by $\Phi(X,Y)$, is the set of Fredholm operators, and their union, $\Phi_\pm(X,Y)$, is the set of semi-Fredholm operators. We shall consider two further subsets of $\mathcal{B}(X,Y)$, called the sets of left Fredholm and right Fredholm operators. These are denoted by $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ and are defined by

$\Phi_l(X,Y) = \{T \in \mathcal{B}(X,Y) : \exists S \in \mathcal{B}(Y,X) \text{ with } ST - I \in \mathcal{F}(X)\}$

and

$\Phi_r(X,Y) = \{T \in \mathcal{B}(X,Y) : \exists S \in \mathcal{B}(Y,X) \text{ with } TS - I \in \mathcal{F}(Y)\}$.

It is well known that $\Phi_l(X,Y)$ is contained in $\Phi_+(X,Y)$ and $\Phi_r(X,Y)$ is contained in $\Phi_-(X,Y)$ and that, for arbitrary Banach spaces, these inclusions are often proper. Indeed, unless Y is a Hilbert space, there always exists a Banach space X such that the inclusions are proper [5]. Our strategy is to reduce the general continuity result to that for left and right Fredholm operators. The latter result is easy and we prove it in this section for the sake of completeness.

Note first that the following index theorem holds just as it does for Fredholm operators on a single space, with the same proof [2 pp.208–9]: For Banach spaces X, Y, Z and operators $T_1 \in \Phi_l(X,Y)$ and $T_2 \in \Phi_l(Y,Z)$ we have $T_2T_1 \in \Phi_l(X,Z)$ and $\text{ind}(T_2T_1) = \text{ind}(T_2) + \text{ind}(T_1)$, with a similar result for right Fredholm operators.

Lemma 1.1. *Let X and Y be Banach spaces. Let $T \in \Phi_l(X,Y)$ and $G \in \mathcal{F}(X,Y)$. Then $T+G \in \Phi_l(X,Y)$ and $\text{ind}(T+G) = \text{ind}(T)$. (A similar result holds for right Fredholm operators).*

Proof: It is obvious that $T+G \in \Phi_l(X,Y)$. Let $P \in \mathcal{B}(Y)$ be any projection of Y onto $G(X)$. Then $(I-P)(T+G) = (I-P)T$ and, since the index of $I-P$ is zero, the result follows from the index theorem. □

Now, since the group of invertible elements of $\mathcal{B}(Y)$ is open, the continuity result for left and right Fredholm operators is a special case of the following proposition:

Proposition 1.2. *Let X and Y be Banach spaces and suppose $T \in \Phi_l(X,Y)$. Suppose $S \in \mathcal{B}(Y,X)$ and $G \in \mathcal{F}(X)$ satisfy $ST-I = G$. Suppose $U \in \mathcal{B}(X,Y)$ is such that $I+US$ is Fredholm of index zero in $\mathcal{B}(Y)$. Then $T+U \in \Phi_l(X,Y)$ and the index of $T+U$ is the same as the index of T.* (The corresponding result with the obvious changes holds for right Fredholm operators).

Proof: Since $T+U = (I+US)T - UG$, the result follows from the index theorem and Lemma 1.1. □

There is a lemma attributed by Banach to Auerbach which states that if X is a finite dimensional normed linear space of dimension n and X^* is its dual, then there exist normalized bases $x_1, \ldots, x_n$ and $f_1, \ldots, f_n$ for X and X^* respectively such that $f_i(x_j) = 1$ if $i = j$ and $f_i(x_j) = 0$ otherwise $(0 \le i,j \le n)$. Ruston's delightfully simple deduction of this result from the compactness of the unit ball can be found in [3, p.200]. The following easy corollaries are given on pages 312–314 of the same work.

Lemma 1.3. *Suppose X is a normed linear space and Y and Z are closed subspaces of X with $\dim(Y) = n$ and $\dim(X/Z) = m$ (m and n finite). Let $\epsilon > 0$. Then there exist projections $P, Q \in \mathcal{B}(X)$ with $P(X) = Y$ and $(I-Q)(X) = Z$ such that $\|P\| \le n$ and $\|Q\| \le m + \epsilon$.*

2. Results

Lemma 2.1. *Let X and Y be Banach spaces. Then the sets* $\{T \in \mathcal{B}(X,Y) : T$ is bounded below and $\mathrm{ind}(T) = -\infty\}$ *and* $\{T \in \mathcal{B}(X,Y) : T$ is surjective and $\mathrm{ind}(T) = \infty\}$ *are open in* $\mathcal{B}(X,Y)$.

Proof: We consider only the first set; the other can be treated similarly. Recall that the set of bounded below operators in $\mathcal{B}(X,Y)$ is open and that such operators are characterized as the closed range operators with zero nullity. Let $T \in \mathcal{B}(X,Y)$ be a bounded below operator and let $\epsilon > 0$ be such that for each $U \in \mathcal{B}(X,Y)$ with $\|U\| < \epsilon$ we have $T + U$ bounded below. Let $(T_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{B}(X,Y)$, converging to T, such that $\|T - T_n\| < \epsilon$ for all $n \in \mathbf{N}$ and suppose $\mathrm{ind}(T_n) > -\infty$ for all $n \in \mathbf{N}$. Then $T_n \in \Phi(X,Y)$ for each $n \in \mathbf{N}$. We must show that $\mathrm{ind}(T) \neq -\infty$. Suppose firstly that the T_n all have the same finite index, $-k$. We shall justify this assumption later. Then $\mathrm{def}(T_n) = k < \infty$ $(n \in \mathbf{N})$, and, by Lemma 1.3, there exists a sequence of projections $(Q_n)_{n\in\mathbf{N}}$ in $\mathcal{F}(Y)$ such that both $T_n(X) = (I - Q_n)(Y)$ and $\|Q_n\| \leq k + \epsilon$ for all $n \in \mathbf{N}$.

Now, for each $n \in \mathbf{N}$, there exists $S_n \in \mathcal{B}(Y,X)$ such that

$$S_nT_n = I \quad \text{and} \quad T_nS_n = I - Q_n.$$

It follows from this that, for each $n \in \mathbf{N}$,

$$\frac{TS_n}{\|S_n\|} = \frac{(T - T_n)S_n}{\|S_n\|} + \frac{I - Q_n}{\|S_n\|}.$$

Since the $I - Q_n$ are uniformly bounded and since T is bounded below, it follows that the S_n are uniformly bounded. So $\|S_n(T - T_n)\| < 1$ for sufficiently large $n \in \mathbf{N}$, and, since $S_nT = I + S_n(T - T_n)$, it follows that S_nT is invertible in $\mathcal{B}(X)$. Hence there exists $S \in \mathcal{B}(Y,X)$ such that $ST = I$. In particular, $T \in \Phi_l(X,Y)$ so that $\mathrm{ind}(T) = -k \neq -\infty$, by Proposition 1.2.

Now our hypothesis that the T_n all have the same index is actually true: Suppose $m, n \in \mathbf{N}$ and let

$$\beta = \sup\{\alpha \in [0,1] : \mathrm{ind}(\alpha T_n + (1-\alpha)T_m) = \mathrm{ind}(T_m)\}.$$

Write $V = \beta T_n + (1-\beta)T_m$. Then V is bounded below, since $\|T + V\| < \epsilon$, and is the limit of a sequence of Fredholm operators each of whose index is $\mathrm{ind}(T_m)$. By what we have just proved, putting V in place of T, we get $\mathrm{ind}(V) = \mathrm{ind}(T_m)$. Now the openness of $\Phi(X,Y)$ and the continuity of the Fredholm index proved in 1.2 ensure that $\beta = 1$ and that $\mathrm{ind}(T_n) = \mathrm{ind}(T_m)$. This completes the proof. □

Theorem 2.2. *$\Phi_\pm(X,Y)$ is an open set and the index is continuous on $\Phi_\pm(X,Y)$.*

Proof: We prove the result for $\Phi_+(X,Y)$. That for $\Phi_-(X,Y)$ can be proved similarly. Suppose then that $T \in \Phi_+(X,Y)$. Since the result for $\Phi(X,Y)$ is contained in Proposition 1.2, we may assume that $\mathrm{ind}(T) = -\infty$. The nullspace Z of T is finite dimensional, so has a complement W in X. Denote by $T_W : W \to Y$ the restriction of T to W. Then T_W is bounded below and also $\mathrm{ind}(T_W) = -\infty$. By Lemma 2.1, there exists $\epsilon > 0$ such that for each $U \in \mathcal{B}(X,Y)$ with $\|U\| < \epsilon$ we have $(T + U)_W \in \mathcal{B}(W,Y)$ bounded below and $\mathrm{ind}(T+U)_W = -\infty$. Since Z is finite dimensional, it follows that $T + U \in \Phi_+(X,Y)$ and $\mathrm{ind}(T + U) = -\infty$ as required. □

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Mícheál Ó Searcóid,
Roinn na Matamaitice,
Coláiste na hOllscoile,
Baile Átha Cliath.

BOUNDARY BEHAVIOUR OF HOLOMORPHIC AND HARMONIC FUNCTIONS*

Stephen J. Gardiner

Abstract: We give below a survey of some recent results concerning the boundary behaviour of holomorphic and harmonic functions. The unifying theme is the role played by the integral condition

$$\int_{-1}^{1} \frac{\phi(t)}{t^2}dt < \infty, \tag{1}$$

where ϕ is a non-negative Lipschitz function.

1. Thin sets

Let Ω be a domain (non-empty, connected open set) in the complex plane $\mathbb{C}$. Recall that a function $u : \Omega \to (-\infty, \infty]$, where $u \not\equiv \infty$, is called *superharmonic* on Ω if u is lower semicontinuous, i.e.,

$$u(z_0) \leq \liminf_{z \to z_0} u(z) \qquad (z_0 \in \Omega), \tag{2}$$

and if

$$u(z_0) \geq \int_0^{2\pi} u(z_0 + re^{i\theta})\frac{d\theta}{2\pi} \text{ when } \{z : |z - z_0| \leq r\} \subset \Omega. \tag{3}$$

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As the definition suggests, such functions need not be continuous. However (assuming that $0 \in \Omega$ for simplicity) we can combine (2), (3) and Fatou's lemma to obtain

$$u(0) \leq \int_0^{2\pi} \liminf_{r\to 0+} u(re^{i\theta})\frac{d\theta}{2\pi} \leq \liminf_{r\to 0+} \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi}$$
$$\leq \limsup_{r\to 0+} \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} \leq u(0),$$

so

$$\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} \to u(0) \qquad (r \to 0+).$$

Thus superharmonic functions possess a certain weak, or "average", continuity property. More specifically, it can be asserted that

$$u(z) \to u(0) \qquad (z \to 0, z \notin E),$$

where the exceptional set E is "thin" at 0. As an upper bound on how much of E exists near 0, we mention that $E\backslash\{0\}$ is contained in an open set whose circular projection, F, onto the interval (0,1) satisfies

$$\sum_{k=1}^{\infty} \frac{k}{-\log|\{t \in F : 2^{-k-1} < t < 2^{-k}\}|} < \infty.$$

(Here $|A|$ denotes the one-dimensional Lebesgue measure of A.) On the other hand, E can be highly dispersed: for example, E can be dense in Ω.

Formally, a set E is called *thin at* 0 if one of the following (equivalent) conditions holds:

(i) there is a superharmonic function u on a neighbourhood of 0 such that

$$\liminf_{z\to 0, z\in E} u(z) > \liminf_{z\to 0} u(z) \; ; \tag{4}$$

(ii) there is a superharmonic function u on a neighbourhood of 0 such that

$$\liminf_{z\to 0, z\in E} \frac{u(z)}{\log 1/|z|} > \liminf_{z\to 0} \frac{u(z)}{\log 1/|z|} \; ;$$

(iii) $\sum_{k=1}^{\infty} kC^*(\{z \in E : 2^{-k-1} \leq |z| \leq 2^{-k}\}) < \infty$

(Wiener's criterion),

where $C^*(A)$ denotes the outer capacity (see [20, Chapter 7]) of a set A with respect to the unit disc.

With regard to condition (i) above, we remark that the right hand side of (4) is equal to $u(0)$. Thus sets E which are thin at 0 are characterized by the property that knowledge of the values of u on $E\backslash\{0\}$ is not sufficient to determine $u(0)$. An account of thin sets can be found in Helms [20, Chapter 10].

There is a corresponding notion of thinness at a boundary point that can be defined by analogy to (ii) above. Let $D_0 = \{x + iy : y > 0\}$ and define

$$P(z) = \frac{1}{\pi}\frac{y}{x^2 + y^2} \qquad (z = x + iy \in D_0).$$

(This is the Poisson kernel for D_0 with pole at 0). A subset E of D_0 is called *minimally thin at* 0 *with respect to* D_0 if there is a positive superharmonic function u on D_0 such that

$$\liminf_{z\to 0, z\in E} \frac{u(z)}{P(z)} > \liminf_{z\to 0, z\in D_0} \frac{u(z)}{P(z)}. \tag{5}$$

Again minimally thin sets may be dense (in D_0), and can only be described in terms of capacities. However, if we are dealing solely with harmonic functions u, the sets E that can arise in (5) are of a more specific nature due to Harnack's inequalities. A precise description of such sets is given below in a reformulation of a result of Beurling [3].

Theorem A. *The following are equivalent conditions on a subset E of D_0:*

(i) *there is a positive harmonic function h on D_0 such that*

$$\liminf_{z\to 0, z\in E} \frac{h(z)}{P(z)} > \liminf_{z\to 0, z\in D_0} \frac{h(z)}{P(z)} ;$$

(ii) *there is a positive number ϵ and a Lipschitz function $\phi : [-1,1] \to [0,\infty)$ such that*

$$\int_{-1}^{1} \frac{\phi(t)}{t^2} dt < \infty \quad \text{and } E \cap \{|z| < \epsilon\} \subseteq \{x+iy : 0 < y < \phi(x)\}.$$

In the following sections we discuss several applications of the above integral condition.

2. The angular derivative problem

In this section D denotes a simply connected domain in $\mathbb{C}$ such that $0 \in \partial D$. Further, f denotes a bijective holomorphic mapping from D_0 to D which has angular limit 0 at 0. (We recall that a function g on D_0 is said to have *angular limit* l at 0 if, for any positive number k,

$$g(z) \to l \qquad (z = x+iy \to 0\ ,\ y > k|x|).)$$

If the derivative f' has an angular limit at 0, this is called the *angular derivative of f at* 0, and is denoted by $f'(0)$. The existence of $f'(0)$ depends on D, but not on the choice of f: that is, if it exists for one such function f then it exists for them all. For further properties of the angular derivative we refer to Pommerenke [21, Chapter 10]. The angular derivative problem is as follows: *give necessary and sufficient geometric conditions on D such that $f'(0)$ exists, and $0 < |f'(0)| < \infty$.*

This problem has a long history and remains unsolved. However, significant progress was made recently by Burdzy [7], using deep probablistic methods. To state his result, we define F_ϵ to be the family of functions $\phi : \mathbb{R} \to \mathbb{R}$ which satisfy the Lipschitz condition $|\phi(x) - \phi(y)| \le |x-y|$ and for which $\{y > \phi(x)\} \cap \{|z| < \epsilon\} \subseteq D \cap \{|z| < \epsilon\}$. Further, let

$$\phi_\epsilon(t) = \inf_{\phi\in F_\epsilon} \phi(t), \quad \phi_\epsilon^+ = \max\{\phi_\epsilon, 0\}, \quad \phi_\epsilon^- = \max\{-\phi_\epsilon, 0\}.$$

Burdzy's theorem is stated below.

Theorem 1. *Suppose that, for some $\epsilon > 0$,*

$$\int_{-1}^{1} \frac{\phi_\epsilon^+(t)}{t^2} dt < \infty.$$

Then $f'(0)$ exists and $0 \le |f'(0)| < \infty$. Further, $f'(0) \ne 0$ if and only if

$$\int_{-1}^{1} \frac{\phi_\epsilon^-(t)}{t^2} dt < \infty.$$

Rodin and Warschawski [22] attempted to prove Theorem 1 by classical means, but were only partly successful: the problem was to find a classical proof of Theorem 2 below, originally proved by Burdzy and Williams [8] using probabilistic methods. This was first achieved by Carroll [11] using an ingenious, but very difficult, argument. Since then two short proofs of the result have been found: one by Sastry [23] based on extremal length arguments, and one by the author [15], based on Beurling's Theorem A. Let $\phi : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and let $D_\phi = \{x+iy : y > \phi(x)\}$.

Theorem 2. *Let $\epsilon > 0$ and let h_ϕ be a positive harmonic function on $D_\phi \cap \{|z| < \epsilon\}$ which continuously vanishes on $\partial D_\phi \cap \{|z| < \epsilon\}$. If*

$$\int_{-1}^{1} \frac{\phi^+(t)}{t^2} dt < \infty \quad \text{and} \quad \int_{-1}^{1} \frac{\phi^-(t)}{t^2} dt = \infty, \tag{6}$$

then $h_\phi(iy)/y \to \infty$ as $y \to 0+$.

The idea of the proof in [15] is to use Theorem A to compare positive harmonic functions $h_0, h_{\phi+}, h_\phi$ on the regions $D_0, D_{\phi+}, D_\phi$ (resp.) which vanish on the boundary, at least near 0. It is easy to see that $h_0(iy)/y$ has a positive limit as $y \to 0+$. The same can be established for $h_{\phi^+}(iy)/y$ using the convergent integral condition in (6). However, the "negative humps" in the boundary of D_ϕ cause $h_\phi(iy)/y$ to diverge to ∞ as $y \to 0+$ because of the divergent integral in (6). This is the tricky part of the proof, as Theorem A does not immediately apply to these "negative humps". See [15] for further details.

3. X-domains

Let U be the unit disc, X be a certain class of holomorphic functions on U, and $\mathcal{H}(U,D)$ be the class of all holomorphic functions $f : U \to D$, where D is some domain in $\mathbb{C}$. In this section we are interested in results of the form: *$f \in X$ for all f in $\mathcal{H}(U,D)$ if and only if D satisfies certain geometric conditions.* For example, X could be the Nevanlinna class $\mathcal{N}$ of holomorphic functions f on U for which

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty \ ;$$

or the Smirnov class $\mathcal{N}^+$ of functions f in $\mathcal{N}$ for which

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \to \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta \qquad (r \to 1-).$$

(Any function f in $\mathcal{N}$ has radial boundary values $f(e^{i\theta})$ almost everywhere.) The following two results are due to Frostman [14] and Ahern and Cohn [1] respectively. A set is called *thin at ∞* if its inversion in the unit circle is thin at 0.

Theorem B. *Let D be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}$ for all f in $\mathcal{H}(U,D)$ if and only if ∂D has positive logarithmic capacity.*

Theorem C. *Let D be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}^+$ for all f in $\mathcal{H}(U,D)$ if and only if $\mathbb{C}\backslash D$ is not thin at ∞.*

We will present two further results of this type. Let h^1 denote the class of harmonic functions on U which can be written as the difference of two positive harmonic functions on U.

Theorem 3. *Let D be a simply connected domain which contains $\{x+iy : x > 0\}$. Then $\Re f \in h^1$ for all f in $\mathcal{H}(U,D)$ if and only if*

$$\int_{-\infty}^{\infty} \frac{\text{dist}\,(iy, \partial D)}{1+y^2} dy < \infty. \qquad (7).$$

We observe that, if $D \subseteq \{x > 0\}$ and $f \in \mathcal{H}(U,D)$, then $\Re f > 0$, so (trivially) $\Re f \in h^1$. Theorem 3 shows precisely how much larger than $\{x > 0\}$ we can allow D to be while still ensuring that $\Re f \in h^1$. The condition (7) is of the same type as (1), after an inversion in the unit circle.

It is easily seen that $\Re f \in h^1$ if and only if $e^f \in \mathcal{N}$. Referring back to Theorems B and C we are led to consider when $e^f \in \mathcal{N}^+$. A subset of $\{x > 0\}$ is called minimally thin at ∞ if its inversion in the unit circle is minimally thin at 0.

Theorem 4. *Let D be as in Theorem 3, suppose (7) holds, and let D_1 be a domain contained in D. Then $e^f \in \mathcal{N}^+$ for all f in $\mathcal{H}(U,D_1)$ if and only if $\{x > 0\}\backslash D_1$ is not minimally thin at ∞ with respect to $\{x > 0\}$.*

Here D_1 is not required to be simply connected. The larger is the set $D\backslash D_1$, the smaller is $\mathcal{H}(U,D_1)$. Theorem 4 describes precisely how large $D\backslash D_1$ must be to ensure that we have the stronger property $e^f \in \mathcal{N}^+$ for all f in $\mathcal{H}(U,D_1)$. It turns out that only $\{x > 0\}\backslash D_1$ is significant. We remark in passing that $\{x > 0\}\backslash D_1$ is not minimally thin at ∞ with respect to $\{x > 0\}$ if and only if the set

$$\left\{(x_1, \cdots, x_4) \in \mathbb{R}^4 : (x_1^2 + x_2^2 + x_3^2)^{1/2} + ix_4 \notin D_1\right\}$$

is not thin at infinity in $\mathbb{R}^4$. Theorems 3 and 4 are proved in [16]. Theorem 3 is related to the angular derivative problem.

4. Sets of determination for harmonic functions

Let $P(\zeta, z)$ denote the Poisson kernel for U with pole ζ; that is,

$$P(\zeta, z) = \frac{1}{2\pi} \cdot \frac{1-|z|^2}{|z-\zeta|^2} \qquad (z \in U, \zeta \in \partial U).$$

If ζ is a fixed point of ∂U, then $P(\zeta, \cdot)$ is a positive harmonic function on U which vanishes on $\partial U \backslash \{\zeta\}$. However, in what follows, we will sometimes fix z and regard $P(\cdot, z)$ as a positive continuous function on ∂U. Let $\mathcal{H}^+$ denote the collection of positive harmonic functions on U. There is a one-to-one correspondence between members h of $\mathcal{H}^+$ and finite Borel measures μ_h on ∂U, given by

$$h(z) = \int_{\partial U} P(\zeta, z) d\mu_h(\zeta) \qquad (z \in U).$$

We consider here two seemingly different types of problem:

(i) given a class A of harmonic functions on U, characterize those subsets E of U such that $\sup_E H = \sup_U H$ for all H in A;

(ii) given a class B of functions on ∂U, characterize those subsets E of U such that any f in B has the form $f = \sum_1^\infty \lambda_k P(\cdot, z_k)$, where the points z_k belong to E.

Surprisingly there is a close relationship between these two types of problem. The key idea in both is that there must be "enough of E" near "appropriate points" of ∂U. We define a set

$$E_{1/2} = \bigcup_{w \in E} \{z : |z-w| < (1-|w|)/2\}$$

and a function

$$E^*_{1/2}(\zeta) = \int_{E_{1/2}} |z-\zeta|^{-2} dx dy \qquad (\zeta \in \partial U),$$

which takes values in $[0, \infty]$. By "enough of E" near ζ we mean $E^*_{1/2}(\zeta) = \infty$.

Theorem 5. *Let $E \subseteq U$. The following are equivalent:*

(i) $\sup_E H = \sup_U H$ *for every H in h^1;*

(ii) $E^*_{1/2}(\zeta) = \infty$ *for every ζ in ∂U;*

(iii) *for every positive continuous function f on ∂U there exist a sequence (λ_k) of positive numbers and a sequence (z_k) of points in E such that*

$$f(\zeta) = \sum_{k=1}^\infty \lambda_k P(\zeta, z_k) \qquad (\zeta \in \partial U). \tag{8}$$

This elegant result is due to Hayman and Lyons [19]. The convergence in (8) is uniform, by Dini's theorem. Alternative proofs and a variety of extensions can be found in [5], [13], [17], [12] and [2]. In particular, [17] contains a short proof based on Beurling's Theorem A (cf. the definition of $E^*_{1/2}(\zeta)$) together with a result which includes the following.

Theorem 6. *Let $E \subseteq U$ and $h \in \mathcal{H}^+$. The following are equivalent:*

(i) $\inf_E H/h = \inf_U H/h$ *for all H in $\mathcal{H}^+$;*

(ii) $E^*_{1/2}(\zeta) = \infty$ *for almost every (μ_h) ζ in ∂U.*

Further, each of the above conditions implies:

(iii) *for every f in $L^1(\mu_h)$ there exist (λ_k) in $\ell_1(\mathbb{C})$ and a sequence (z_k) of points in E such that*

$$f = \sum_{k=1}^\infty \lambda_k P(\cdot, z_k)/h(z_k) \tag{9}$$

(convergence in the sense of $L^1(\mu_h)$), and

$$\|f\|_{L^1(\mu_h)} = \inf\{\Sigma|\lambda_k| : (9) \text{ holds for some } (z_k) \text{ in } E\}.$$

If $h \equiv 1$, then (cf. Bonsall [4]) (iii) above is actually *equivalent* to (i), (ii) and:

(ii′) *for almost every (Lebesgue) ζ in ∂U, there is a sequence of points in E which converges to ζ within some angle.*

5. Better-than-angular limits

Sections 2 and 3 illustrated the relevance of (1) to boundary distortion. In section 4, divergence of the integral in (1) was related to having "enough" of E to determine suprema/infima of harmonic functions on U, or to achieve representation of functions on ∂U in terms of Poisson kernels. Finally, in this section, we discuss the role of (1) in describing approach regions for boundary behaviour of holomorphic and harmonic functions.

Many results in function theory state that functions (in D_0, say) have angular limits almost everywhere on ∂D_0, or on a subset of ∂D_0 of positive Lebesgue measure. We will now point out that rather more can be asserted. Let Φ denote the class of Lipschitz functions $\phi : \mathbb{R} \to [0, \infty)$ such that (1) holds. A function g on D_0 is said to have *Φ-limit l at $t \in \mathbb{R}$* if there exists ϕ in Φ such that

$$g(z) \to l \qquad (z \to t, \ y > \phi(x - t)).$$

It is easy to see that the existence of a Φ-limit at t implies that g has an angular limit at t, but not conversely.

Theorem 7. *Let u be a harmonic function on D_0 such that, for every t in E (where $E \subseteq \mathbb{R}$), there is an angle with vertex at t in which u is bounded below. Then u has (finite) Φ-limits at almost every (Lebesgue) t in E.*

The existence of angular limits under the above hypothesis is due to Calderón [9] and Carleson [10]. Brelot and Doob [6] showed that u must have minimal fine limits at almost every point t in E. However, the latter result does not immediately combine with Theorem A to yield Theorem 7, since u need not be positive on D_0. The proof of Theorem 7 can be found in [15]. An example of its application occurs in [18]. The conclusion of the theorem clearly remains true if u is a holomorphic function on D_0 which, for each t in E, is bounded in an angle with vertex at t.

NOTE. The results in this paper which concern harmonic functions have natural analogues in $\mathbb{R}^n (n \geq 3)$. Details can be found in the appropriate references.

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INTERNAL FORCING AXIOMS: MARTIN'S AXIOM AND THE PROPER FORCING AXIOM

Dedicated to the memory of Alan H. Mekler.

Eoin Coleman

In the course of the last twenty-five years research in the combinatorics of partially ordered sets has resulted in the discovery of new set-theoretic hypotheses — sometimes dubbed internal forcing axioms. This elementary article presents in section 1 the simplest of these (Martin's Axiom). In section 2 we look at some applications (the completeness of the category ideal, Lusin sets, Q-sets, problems of Moore, Alexandroff, Suslin, Whitehead and Kaplansky). Finally in section 3 we deal briefly with the Proper Forcing Axiom, a powerful generalization of Martin's Axiom. We've collected the relevant references in an annotated bibliography in section 4, rather than in the body of the text.

We try to show concretely how internal forcing axioms work (giving complete proofs whenever feasible), stressing the resemblance to the classical diagonal arguments of Baire and Cantor. In our choice of applications we seek to underline the fact that mathematical conjectures having no apparent set-theoretic reference may depend for their resolution on axioms beyond those of ordinary set theory. To put it another way, there are at least three truth values in mathematics: true, false, and independent of ordinary set theory.

Section 1: Forcing

Internal forcing axioms are about forcings. Let us recall that a *forcing* is simply a partial order, i.e. a pair $\mathbf{P} = (P, \leq)$ such that P is a non-empty set, $\leq$ is a reflexive antisymmetric transitive binary relation on P (so, for all $p, q, r \in P$, (i) $p \le p$, (ii) if $p \le q$ and $q \le p$, then $p = q$, and (iii) if $p \le q$ and $q \le r$, then $p \le r$). Elements of P are called *conditions*. Conditions p and q are *compatible* iff they have a common upper bound in P, i.e. $(\exists r \in P)(p \le r$ and $q \le r)$; otherwise p and q are *incompatible*. A subset $D \subseteq P$ is *dense in* $\mathbf{P}$ iff $(\forall p \in P)(\exists r \in D)(p \le r)$. A non-empty subset G of P is a *filter in* $\mathbf{P}$ iff $(\forall p, q \in G)(\exists r \in G)(p \le r$ and $q \le r)$ and $(\forall p \in P)(\forall q \in G)($if $p \le q$, then $p \in G)$. Finally if $\mathcal{D}$ is a family of dense sets in $\mathbf{P}$, we say that a filter G in $\mathbf{P}$ is *$\mathcal{D}$-generic* iff for every $D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

To sort out these definitions, consider the following situation.

Example 1.1: Adding a Cohen real. Let P be the set $\{f : f$ is a function from a finite subset of $\mathbf{N}$ to $\{0,1\}\}$ and define a partial ordering on P by $f \le g$ iff g extends f, i.e. $\operatorname{dom} f \subseteq \operatorname{dom} g$ and $g \restriction \operatorname{dom} f = f$. Certainly $\mathbf{P} = (P, \le)$ is a forcing. Conditions f and g are compatible iff they agree on $\operatorname{dom} f \cap \operatorname{dom} g$, in which case the union $f \cup g$ is a condition extending f and g. So if G is a filter in $\mathbf{P}$, then $\bigcup G = \bigcup\{f : f \in G\}$ is a function from a subset of $\mathbf{N}$ to $\{0,1\}$, since the union of compatible functions is itself a function. Note also that if $f \in G$ and $n \in \operatorname{dom} f$, then $(\bigcup G)(n) = f(n)$. Examples of dense sets are the sets $C_m = \{g \in P : m \in \operatorname{dom} g\}$ for each $m \in \mathbf{N}$: given any $f \in P$, either $f \in C_m$, or $m \notin \operatorname{dom} f$ and then $g = f \cup (m, 0)$ belongs to C_m and $f \le g$. Observe that the dense sets which G intersects determine to some extent the function $\bigcup G$: for example, if $G \cap C_m \neq \emptyset$, then $m \in \operatorname{dom} \bigcup G$. So if G is $\mathcal{C}$-generic where $\mathcal{C} = \{C_m : m \in \mathbf{N}\}$, then $\bigcup G$ is a function from (all of) $\mathbf{N}$ to $\{0,1\}$. If $\mathcal{D} \supseteq \mathcal{C}$ and G is $\mathcal{D}$-generic, then $\bigcup G$ is called a *Cohen real*. Note that a Cohen real does not belong to P, since its domain is the infinite set $\mathbf{N}$.

Internal forcing axioms are putatively consistent answers to the natural question: for which forcings $\mathbf{P}$ and families $\mathcal{D}$ of dense sets in $\mathbf{P}$ does there exist a $\mathcal{D}$-generic filter G in $\mathbf{P}$? The first and weakest internal forcing axiom is a very easy Cantorian diagonal argument.

Proposition 1.2. *If $\mathbf{P}$ is a forcing and $\mathcal{D}$ is a countable family of dense sets in $\mathbf{P}$, then there is a $\mathcal{D}$-generic filter G in $\mathbf{P}$.*

Proof: Enumerate $\mathcal{D}$ as $\{D_n : n \in \mathbf{N}\}$ and by induction on n choose p_n such that $p_0 \in P$ and for $n \ge 1$, $p_n \in D_{n-1}$ and $p_{n-1} \le p_n$ (possible since D_{n-1} is dense in $\mathbf{P}$). Now let $G = \{q \in P : (\exists n \in \mathbf{N})(q \le p_n)\}$.

We'll apply this to prove a very well-known theorem.

Corollary 1.3 (The Baire category theorem). *If X is a compact Hausdorff space (or a complete metric space) and A_n is a (topologically) dense open subset of X for $n \in \mathbf{N}$, then $\bigcap\{A_n : n \in \mathbf{N}\}$ is non-empty.*

Proof: Let P be the set $\{p \subseteq X : p$ is a non-empty open set$\}$ and define $p \le q$ iff $q \subseteq p$. For $n \in \mathbf{N}$ the set $D_n = \{p \in P : \mathrm{Cl}(p) \subseteq A_n\}$ is dense in $\mathbf{P}$: given q in P, we know $A_n \cap q \neq \emptyset$, so since X is regular there is $p \in P$ such that $\mathrm{Cl}(p) \subseteq A_n \cap q$; now $p \in D_n$ and $q \le p$. Proposition 1.2 yields a filter G which intersects each D_n non-trivially. Let A be $\bigcap\{\mathrm{Cl}(p) : p \in G\}$. Clearly $A \subseteq \bigcap\{A_n : n \in \mathbf{N}\}$ since $G \cap D_n \neq \emptyset$. Note also that for each finite $F \subseteq G$, $\bigcap\{\mathrm{Cl}(p) : p \in F\}$ is non-empty: G is a filter, so there is $r \in G$ $(\forall p \in F)(p \le r)$ and so $\emptyset \neq r \subseteq \bigcap\{\mathrm{Cl}(p) : p \in F\}$. Now since X is compact, it follows that A is non-empty.

As it stands, Proposition 1.2 is the best one can do. If $\mathcal{D}$ is uncountable, the conclusion does not necessarily hold.

Proposition 1.4. *There is a forcing Q and an uncountable family $\mathcal{R}$ of dense sets for which there is no generic $\mathcal{R}$-filter.*

Proof: Let I be an uncountable set, let Q be $\{f : f$ is a function from a finite subset of $\mathbf{N}$ to $I\}$, and define $f \le g$ iff g extends f. For $i \in I$, the set $R_i = \{f \in Q : i \in \operatorname{range} f\}$ is dense in Q. Taking $\mathcal{R} = \{R_i : i \in I\}$, we note that if G were an $\mathcal{R}$-generic filter, then $\bigcup G$ would be a function from a (countable) subset of $\mathbf{N}$ onto the uncountable set I — an impossibility.

We can make precise an important difference between situations 1.1 and 1.4 by considering the sizes of the sets of pairwise incompatible conditions in the respective forcings.

Definition 1.5. Suppose $\mathbf{P}$ is a forcing.
(1) An *antichain* in $\mathbf{P}$ is a set $A \subseteq P$ of pairwise incompatible conditions.
(2) We say that $\mathbf{P}$ has the *countable chain condition* ($\mathbf{P}$ is c.c.c.) iff every antichain in $\mathbf{P}$ is countable.
(3) A subset C of P is a *chain* in $\mathbf{P}$ iff $(\forall p, q \in C)(p \leq q$ or $q \leq p)$.

Some authors refer to (2) as the countable antichain condition.

Thus the Cohen forcing $\mathbf{P}$ of 1.1 is c.c.c. trivially, since P is itself a countable set, whereas in 1.4 Q is not c.c.c., since $A = \{f_i : i \in I\}$ is an uncountable antichain, where $f_i(0) = i$ for $i \in I$. By restricting attention to c.c.c. forcings, we avoid the counterexample of 1.4 at least, and it makes sense to reformulate Proposition 1.2 for c.c.c. forcings and uncountable families of dense sets.

Definition 1.6. We let MA_κ abbreviate the hypothesis: if $\mathbf{P}$ is a c.c.c. forcing, $\mathcal{D}$ is a family of dense sets in $\mathbf{P}$ and $\mathcal{D}$ has cardinality at most κ, then there is a $\mathcal{D}$-generic filter G in $\mathbf{P}$.

Just to clear up some notation: we use $\kappa, \lambda, \ldots$ to denote infinite cardinals; the first infinite cardinal is $\aleph_0$; the first uncountable cardinal is $\aleph_1$. For a set X, $|X|$ is the cardinality of X, $P(X)$ is the power set of X. The cardinal $2^{|X|}$ is $|\{f : f$ is a function from X to $\{0,1\}\}|$; λ^+ is the least cardinal greater than λ. For example, $\aleph_1 = \aleph_0^+$, $\aleph_0 = |\mathbf{N}|$, $2^{\aleph_0} = |\mathbf{R}|$, and $2^{|X|} = |P(X)|$ (identifying subsets of X with their characteristic functions).

For each infinite cardinal κ we obtain a version of 1.2 for c.c.c. forcings and families of dense subsets of cardinality at most κ. Some are obviously true; some are false.

Proposition 1.7. (1) $\mathrm{MA}_{\aleph_0}$ *is true.* (2) MA_κ *implies* $\kappa < 2^{\aleph_0}$. (3) MA_λ *is false for every* $\lambda \geq 2^{\aleph_0}$.

Proof: Proposition 1.2 clearly implies 1.7 (1). Part (3) follows from (2). For (2), we show that there is no mapping F from κ onto the set ${}^{\mathbf{N}}2 = \{f : f$ is a function from $\mathbf{N}$ to $\{0,1\}\}$. Suppose that F maps κ to ${}^{\mathbf{N}}2$. Let $H = \mathrm{range}\, F$. For each $h \in H$, let $R_h = \{f \in P : (\exists\, n \in \mathrm{dom}\, f)(f(n) = 1 - h(n))\}$, where $\mathbf{P}$ is the Cohen forcing of Example 1.1. Note that R_h is dense in $\mathbf{P}$. As in 1.1, let $C_m = \{f \in P : m \in \mathrm{dom}\, f\}$. Now $\mathcal{D} = \{C_m, R_h : m \in \mathbf{N}, h \in H\}$is a family of dense sets in $\mathbf{P}$ and $\mathcal{D}$ has cardinality at most κ (recall $\kappa + \aleph_0 = \kappa$ since κ is infinite). By MA_κ there is a $\mathcal{D}$-generic filter G in $\mathbf{P}$. The Cohen real $\bigcup G$ belongs to ${}^{\mathbf{N}}2$ (since $G \cap C_m \neq \emptyset\ \forall m \in \mathbf{N}$), but does not belong to H (since for each $h \in H$, $G \cap R_h \neq \emptyset$, so $(\exists\, n \in \mathbf{N})(\bigcup G(n) = 1 - h(n))$, giving $\bigcup G \neq h$). Thus F is not onto.

Remark that letting $\kappa = \aleph_0$ in (2) and using (1), one obtains Cantor's theorem: $2^{\aleph_0} > \aleph_0$. The original diagonal argument runs as follows: if $\{h_n : n \in \mathbf{N}\} \subseteq {}^{\mathbf{N}}2$, then the function g defined by $g(n) = 1 - h_n(n)$ for $n \in \mathbf{N}$ belongs to ${}^{\mathbf{N}}2$ but differs in the nth place from each h_n. In the argument from 1.7 (1) (2), one finds the required function g by considering the c.c.c. forcing consisting of the finite approximations to g and defining appropriate dense sets.

Guided by the information in Proposition 1.7 we write down Martin's Axiom.

Definition 1.8. *Martin's Axiom* MA is the hypothesis $(\forall \kappa < 2^{\aleph_0})(\mathrm{MA}_\kappa$ holds).

From the definition and Proposition 1.7 we obtain immediately:

Corollary 1.9. (1) *The Continuum Hypothesis* CH *(*$2^{\aleph_0} = \aleph_1$*) implies* MA. (2) CH *implies that* $\mathrm{MA}_{\aleph_1}$ *is false.*

Of course if CH holds, then MA is just $\mathrm{MA}_{\aleph_0}$ and of little interest since we can prove the stronger result 1.2. For this reason MA is often taken to mean MA and $\neg$CH ($2^{\aleph_0} > \aleph_1$). In this connection, Solovay and Tennenbaum established the following relative consistency result, which we shall discuss in section 3.

Theorem 1.10. CON(ZFC + MA + $\neg$CH)*, i.e. the system of axioms of ordinary set theory and* MA *and* $\neg$CH *is consistent.*

In other words, if no contradiction can be deduced from ZFC (the axioms of ordinary set theory), then none can be deduced from ZFC + MA + $\neg$CH. We'll often use the equivalent semantic formulation: there is a set-theoretic universe (a model of ZFC) in which $2^{\aleph_0} > \aleph_1$ and MA holds. It follows immediately from 1.9 (2) and 1.10 that $MA_{\aleph_1}$ is independent of ordinary set theory, i.e. $MA_{\aleph_1}$ can neither be proved nor refuted from ZFC.

We finish the proofs for this section by showing that MA_κ is equivalent to the seemingly weaker axiom MA_κ^-: if $\mathbf{P}$ is a c.c.c. forcing of cardinality at most κ, $\mathcal{D}$ is a family of dense sets in $\mathbf{P}$ and $\mathcal{D}$ has cardinality at most κ, then there is a $\mathcal{D}$-generic filter G in $\mathbf{P}$.

Proposition 1.11. MA_κ^- *implies* MA_κ.

Proof: Given a family $\mathcal{D}$ of dense sets in an arbitrary forcing $\mathbf{P}$ we find a suitable subforcing $\mathbf{Q}$ of cardinality at most κ as follows. Let c be a (partial) function from $P \times P$ to P defined thus: if p and q are compatible, $c(p,q)$ is a common upper bound (otherwise $c(p,q)$ is not defined). For each $D \in \mathcal{D}$, let $c_D : P \to D$ be defined by $c_D(p) \in D$, $p \leq c_D(p)$. Now let Q be a non-empty subset of P of cardinality at most κ closed under the functions c and c_D for $D \in \mathcal{D}$. Easily $\mathbf{Q} = (Q, \leq\restriction Q)$ is a c.c.c. forcing of cardinality at most κ, and for $D \in \mathcal{D}$, $Q \cap D$ is dense in Q. So by MA_κ^-, there is a filter H in Q intersecting every $Q \cap D$. The filter $G = \{p \in P : (\exists q \in H)(p \leq q)\}$ is now $\mathcal{D}$-generic in $\mathbf{P}$.

And to make explicit the connection between the internal forcing axioms of this section and the Baire category theorem, we should point out that 1.3 implies 1.2 and MA_κ is equivalent to the topological hypothesis: if X is a c.c.c. compact Hausdorff space, then the intersection of at most κ dense open subsets of X is non-empty. (Remember that X is c.c.c. means that every collection of pairwise disjoint non-empty sets is countable.)

Section 2: Applications of MA

In this section we prove some easy independence results (Lusin sets, Q-sets) and mention some further applications of MA. Our first aim is to study the effect of MA on the real numbers: what kinds of subsets does $\mathbf{R}$ have?

Recall some Baire category terminology: a subset N of a space X is *nowhere dense* iff $X \setminus \mathrm{Cl}(N)$ is a dense open set (equivalently, $\mathrm{Int}(\mathrm{Cl}(N)) = \emptyset$); a subset F of X is *of first category* iff F is a countable union of nowhere dense sets in X.

Theorem 2.1. *Assume* MA. *Suppose X is a second countable space. If $\mathcal{F}$ is a family of nowhere dense sets and $\mathcal{F}$ has cardinality $\kappa < 2^{\aleph_0}$, then $\bigcup \mathcal{F}$ is of first category. For example,* MA *implies that every set of reals of cardinality less than $2^{\aleph_0}$ is of first category, and the category ideal on $\mathbf{R}$ is complete: the union of fewer than $2^{\aleph_0}$ first category subsets of $\mathbf{R}$ is of first category.*

To prove 2.1 we need a useful combinatorial lemma about $P(\mathbf{N})$.

Lemma 2.2. *Assume* MA_κ. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of $\mathbf{N}$, $\mathcal{A}$ and $\mathcal{B}$ have cardinality at most κ, and if $A_1, \ldots, A_n \in \mathcal{A}$, $B \in \mathcal{B}$, then $B \setminus (A_1 \cup \ldots A_n)$ is infinite. Then there exists $C \subseteq \mathbf{N}$ such that $C \cap A$ is finite and $C \cap B$ is infinite for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.*

Proof: Write $\mathcal{A} = \{A_i : i \in I\}$, $\mathcal{B} = \{B_i : i \in I\}$ where I has cardinality κ (allowing repetitions if necessary). Define P to be the following set: $\{(h,a) : h$ is a finite subset of I and a is a finite subset of $\mathbf{N}\}$; say $(h,a) \leq (k,b)$ iff $h \subseteq k$, $a \subseteq b$ and $(b \setminus a) \cap (\bigcup_{i \in h} A_i) = \emptyset$. It is straightforward to check that $\mathbf{P} = (P, \leq)$ is a forcing. To see that $\mathbf{P}$ is c.c.c., note that (h,a) and (k,a) are compatible for any h and k, so if $W \subseteq P$ is an antichain in $\mathbf{P}$, then W is countable (since there are only countably many possibilities for the second components of elements of W). It's easy to check that the sets $D_i = \{(h,a) : i \in h\}$ and $E_{i,n} = \{(h,a) : |a \cap B_i| > n\}$ are dense in $\mathbf{P}$.

Now apply MA_κ to get a filter G intersecting each member of the family $\mathcal{L} = \{D_i, E_{i,n} : i \in I,\ n \in \mathbf{N}\}$ ($\mathcal{L}$ has cardinality at most κ). We'll complete the proof by showing that $C = \bigcup\{a : (\exists h)[(h,a) \in G]\}$ is as required. Fix $i \in I$. Since $G \cap E_{i,n} \neq \emptyset$, it follows that $|C \cap B_i| > n$ for each $n \in \mathbf{N}$ and so $C \cap B_i$ is infinite. Also $G \cap D_i \neq \emptyset$, so take $(h,a) \in G \cap D_i$ and note that

$C \cap A_i \subseteq a$ (if $(k,b) \in G$, then $(b \setminus a) \cap A_i \neq \emptyset$ since (h,a) and (k,b) are compatible); so $C \cap A_i$ is finite.

Lemma 2.2 for countable collections $\mathcal{A}$ and $\mathcal{B}$ is a simple exercise (in ZFC) which does not require any diagonalization. Let's go back now to the proof of Theorem 2.1.

Well, X is second countable, so one can choose a listing $\{U_n : n \in \mathbf{N}\}$ of a countable basis for the topology on X in which each non-empty basic open set is listed infinitely many times. Let $B_n = \{m \in \mathbf{N} : U_m \subseteq U_n\}$; for $F \in \mathcal{F}$, let $A_F = \{m \in \mathbf{N} : U_m \cap F \neq \emptyset\}$; take $\mathcal{A} = \{A_F : F \in \mathcal{F}\}$, $\mathcal{B} = \{B_n : n \in \mathbf{N}\}$. To see that $\mathcal{A}$ and $\mathcal{B}$ satisfy the hypotheses in 2.2 for $\kappa = \max(|\mathcal{F}|, \aleph_0)$, remember that a finite union of nowhere dense sets is nowhere dense and that every basic open set is listed infinitely many times. Apply MA_κ to find C as in Lemma 2.2. Let $R_n = \bigcup\{U_m : m \in C$ and $m \geq n\}$. R_n is a dense open subset of X: given U_k, choose $m \in C \cap B_k$, $m \geq n$; so $U_m \subseteq U_k$ and $U_m \subseteq R_n$. Finally let M_n be the closed nowhere dense set $X \setminus R_n$. It'll suffice to show that $\bigcup \mathcal{F} \subseteq \bigcup_{n \in \mathbf{N}} M_n$. For $F \in \mathcal{F}$, $C \cap A_F$ is finite; pick $n \in C \setminus A_F$, $n > \max(C \cap A_F)$, then for every $m \in C$, $m \geq n$ gives $U_m \cap F = \emptyset$, so $F \subseteq \bigcap\{X \setminus U_m : m \in C, m \geq n\} = M_n$.

In passing, we note that a similar result holds replacing sets of first category by sets of Lebesgue measure zero.

From 1.3 and 2.1 we obtain an independence result. Let $C(\aleph_1)$ abbreviate the assertion: if $A \subseteq \mathbf{R}$ has cardinality $\aleph_1$, then A is of first category. We conclude that $C(\aleph_1)$ is independent of ordinary set theory: if CH holds, then $C(\aleph_1)$ is false ($\mathbf{R}$ is a counterexample (by 1.3)); if MA + $\neg$CH holds, then $C(\aleph_1)$ is true (by 2.1).

Before going on to Lusin sets, we need to count the subsets of $\mathbf{R}$.

Proposition 2.3. *The following collections have cardinality exactly $2^{\aleph_0}$: (1) the open sets of reals; (2) the closed sets; (3) the closed nowhere dense sets.*

Proof: Ad (1): Every open set can be expressed as a countable union of open intervals with rational endpoints. There are countably many such intervals, so there are at most $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ possible choices for open sets of reals. Easily there are at least $2^{\aleph_0}$ open sets.

Part (2) follows from (1), closed sets of reals being exactly the complements of open sets; part (3) is immediate from (2).

Lusin sets are a little more obscure than uncountable sets of first category:

Definition 2.4. A subset K of $\mathbf{R}$ is called a *Lusin set* iff
(i) K is uncountable and
(ii) whenever $F \subseteq \mathbf{R}$ is of first category, then $K \cap F$ is countable.

Lusin sets (discovered of course by Mahlo) are rather unusual: with regard to category, they are not small (no uncountable subset of K is of first category); with regard to Lebesgue measure, they are very small indeed (recall that for every positive ϵ there is a closed nowhere dense set N such that $\mathbf{R} \setminus N$ has measure less than ϵ). But are there any Lusin sets? Well, it depends.

Theorem 2.5. (1) CH *implies that there is a Lusin set.*
(2) MA + $\neg$CH *implies that there are no Lusin sets.*

Proof: Ad (1): By 2.3 and CH we can list all the closed nowhere dense sets in a list $\{N_\alpha : \alpha < \aleph_1\}$. Define $K = \{r_\alpha : \alpha < \aleph_1\}$ by transfinite induction on $\alpha < \aleph_1$. Given $\{r_\beta : \beta < \alpha\}$ note that $M_\alpha = \bigcup\{N_\beta : \beta < \alpha\} \cup \{r_\beta : \beta < \alpha\}$ is of first category (since α is a countable ordinal), so by 1.3 one can find $r_\alpha \in \mathbf{R} \setminus M_\alpha$. By construction, K is a Lusin set: if F is of first category, then for some $\alpha < \aleph_1$, $F \subseteq M_\alpha$ and so $K \cap F \subseteq \{r_\beta : \beta < \alpha\}$.

Ad (2): Supposing contrariwise that K is Lusin let $F \subseteq K$ be a subset of cardinality $\aleph_1$. By 2.1, F is of first category, being the union of its singleton sets — in contradiction to 2.4 (ii). So K doesn't exist.

The Q-sets which we define next occur naturally in the study of Moore spaces. We'll explain why after the definition and some basic facts.

Definition 2.6. A set $A \subseteq \mathbf{R}$ is a *Q-set* iff every subset of A is a relative F_σ (i.e. a countable union of closed sets in the subspace topology on A). For example, every countable set is a Q-set. Are there any uncountable Q-sets?

Proposition 2.7. (1) *If $2^{\aleph_0} < 2^\kappa$, then there are no Q-sets of cardinality κ.*
(2) *If A is a Q-set, then A has cardinality less than $2^{\aleph_0}$. In particular,* CH *implies that every Q-set is countable.*

Proof: Part (2) is a consequence of (1), noting that $\lambda < 2^\lambda$ and taking $\lambda = 2^{\aleph_0}$. As regards (1), suppose that B has cardinality κ. By 2.3 there are at most $2^{\aleph_0}$ relatively closed subsets of B, so there are at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ relative F_σ's of B. However, $|P(B)| = 2^{|B|} = 2^\kappa > 2^{\aleph_0}$, so some subset of B is not a relative F_σ, i.e. B is not a Q-set.

Theorem 2.8. *Assume* MA. (1) *Every set of reals of cardinality less than $2^{\aleph_0}$ is a Q-set.*
(2) For $\aleph_0 < \kappa < 2^{\aleph_0}$, $2^\kappa = 2^{\aleph_0}$. (3) $2^{\aleph_0}$ *is a regular cardinal.*

Proof: Part (1) is similar to 2.1 and we give just a sketch. Suppose $X \subseteq A \subseteq \mathbf{R}$ and A has cardinality $\kappa < 2^{\aleph_0}$. We show X is a relative F_σ. WLOG X is a non-empty proper subset of A. Choose a countable open basis $\{V_n : n \in \mathbf{N}\}$ for $\mathbf{R}$ such that no two different reals belong to the intersection of infinitely many V_n. Let $O_x = \{n \in \mathbf{N} : x \in V_n\}$ and note that $\mathcal{A} = \{O_x : x \in X\}$ and $\mathcal{B} = \{O_x : x \in A \setminus X\}$ satisfy the hypotheses of 2.2. Using C from 2.2, the open sets $G_n = \bigcup\{V_k : k \in C$ and $k \geq n\}$, the closed sets $F_n = \mathbf{R} \setminus G_n$, one verifies that $X \subseteq \bigcup\{F_n : n \in \mathbf{N}\}$, $A \setminus X \subseteq \bigcup\{G_n : n \in \mathbf{N}\}$ and so X is a relative F_σ.

Ad (2): Let $B \subseteq \mathbf{R}$ have infinite cardinality $\kappa < 2^{\aleph_0}$. By part (1), B is a Q-set, hence by 2.7 (1), $2^{\aleph_0} = 2^\kappa$. Ad (3): Since $\mathbf{R}$ has cardinality $2^{\aleph_0}$, we work with $\mathbf{R}$. If $\mathbf{R} = \bigcup\{A_i : i < \lambda\}$ where $|A_i| < 2^{\aleph_0}$, then by 2.1 each A_i is of first category and so by 2.1 again $\lambda \geq 2^{\aleph_0}$. (The reader familiar with Koenig's Lemma will deduce part (3) immediately from part (2).)

Thus MA + ¬CH implies that there are uncountable Q-sets. Taken in conjunction with 2.7 (2) this means that the existence of an uncountable Q-set is independent of ZFC.

From 2.8 (3) it also follows that ¬CH does not imply MA, since there are set-theoretic universes in which $2^{\aleph_0}$ is not a regular cardinal.

Uncountable Q-sets are related to the Normal Moore Space Conjecture (NMSC) which states that all normal Moore spaces are metrizable. A space is *normal* iff for all disjoint closed sets A and B there are disjoint open sets U and V, $A \subseteq U$, $B \subseteq V$. A *Moore space* is a regular space X with a sequence of open covers $\{\mathcal{G}_n : n \in \mathbf{N}\}$ such that for each $x \in X$ and open U with $x \in U$, there is $n \in \mathbf{N}$ such that $\bigcup\{G \in \mathcal{G}_n : x \in G\} \subseteq U$. Examples of normal non-metrizable Moore spaces can be obtained in the following way.

Example 2.9. For this we take an uncountable set $B \subseteq \mathbf{R}$. Let $M(B)$ be the set $B \cup \{(x, y) \in \mathbf{R}^2 : y > 0\}$; the neighbourhoods of $b \in B$ are the bubbles at b, i.e. $\{b\} \cup \mathrm{Int}(D)$ where D runs over the discs in the upper half-plane tangent to the x-axis at $(b, 0)$; the neighbourhoods of (x, y) are the usual Euclidean ones. $M(B)$ is called the Moore space derived from B and is a separable non-metrizable Moore space. It turns out that $M(B)$ is normal iff B is a Q-set. It is also known that the existence of an uncountable Q-set is equivalent to the existence of a separable normal non-metrizable Moore space. So MA + ¬CH implies that NMSC is false, even in the separable case. Of course this leaves open the question whether the falsity of NMSC follows just from ordinary set theory. The resolution of this issue is a little different from the independence results we've considered so far. It involves so-called large cardinal axioms, axioms which roughly speaking assert the existence of cardinals so large that they cannot be shown to exist on the basis of ordinary set theory. We state the result, omitting the technical definitions and details:

Theorem 2.10. (1) *If* NMSC *holds, then there is an inner model of* ZFC *with a measurable cardinal.* (2) *The Product Measure Extension Axiom (*PMEA*) implies* NMSC. (3) *If* ZFC + *"there is a strongly compact cardinal" is consistent, then* ZFC + PMEA *is consistent, and so* ZFC + NMSC *is consistent.*

Before leaving metrizability questions, let us mention an application of MA in the study of manifolds. Taking a manifold to be a connected regular Hausdorff space M for which there is a positive integer n such that each point of M has a neighbourhood homeomorphic to $\mathbf{R}^n$, one can prove:

Theorem 2.11. (1) *Assume* MA + ¬CH. *Then every perfectly normal manifold is metrizable.*
(2) *There is a set-theoretic universe* L *in which there exists a perfectly normal non-metrizable manifold.*

Thus again the answer to a query of Alexandroff is independent of ZFC.

Our next application concerns the uniqueness of the real line $(\mathbf{R}, \leq)$. Suslin's Hypothesis claims that there are no Suslin trees. Recall that a *Suslin tree* is an uncountable c.c.c. partial order $\mathbf{T} = (T, \leq)$ satisfying: (a) $(\forall t \in T)\,\mathrm{Pred}(t) = \{s \in T : s < t\}$ is a chain which is well-ordered, i.e. every non-empty subset of $\mathrm{Pred}(t)$ has a $<$-least element; (b) T has no uncountable chains; (c) $(\forall t \in T)\,\mathrm{Suc}(t) = \{s \in T : t < s\}$ is uncountable. The study of Suslin's Hypothesis led to the discovery of Martin's Axiom.

Theorem 2.12. $\mathrm{MA}_{\aleph_1}$ *implies* SH: *there are no Suslin trees.*

Proof: Suppose for a contradiction that T is a Suslin tree. By (a) and (b), $\mathrm{Pred}(t)$ is order-isomorphic to a countable ordinal $h(t)$, the height of t in T, so (c) implies that the set $D_\alpha = \{t \in T : h(t) \geq \alpha\}$ is dense in T for each ordinal $\alpha < \aleph_1$. Apply $\mathrm{MA}_{\aleph_1}$ to find a filter G in T intersecting each D_α non-trivially. Now G is an uncountable chain in T, contradicting (b).

It is consistent with ordinary set theory to assume that SH is false. For example, in L (the smallest transitive set-theoretic universe containing all the ordinals) there is a Suslin tree.

Most mathematics students learn (in a possibly different terminology) that if $(S, \leq)$ is a separable, uncountable, unbounded, Dedekind-complete, dense linear order, then $(S, \leq)$ is order isomorphic to the real line $(\mathbf{R}, \leq)$ (just recall the well-known back-and-forth argument of Cantor characterizing the rational line $(\mathbf{Q}, \leq)$). Suslin's Hypothesis is equivalent to the assertion that separability can be replaced by the condition that every collection of pairwise disjoint open intervals in the linear order is countable.

Finally we turn to two famous applications of MA in algebra and analysis. We say that an infinite abelian group A is a *free group* iff A has a linearly independent set of generators; we say that A is a *W-group* iff for every surjective homomorphism $\pi : B \to A$ with kernel $\mathbf{Z}$ there is a homomorphism $\phi : A \to B$ such that $\pi(\phi(a)) = a$ for all $a \in A$ (in other words $\mathrm{Ext}(A, \mathbf{Z}) = 0$). For example, every free group is a W-group. Whitehead asked: is every W-group free? Shelah showed that $\mathrm{MA}_{\aleph_1}$ implies the existence of a non-free W-group. He was also able to prove that in L every W-group is free. So the Whitehead problem is independent of ZFC. It is remarkable that the concepts involved in his research yield, via trees, considerable information on NMSC.

Let's conclude this section with an automatic continuity problem in analysis. Recall that $C[0,1]$ is the commutative Banach algebra of continuous functions on the closed unit interval. Kaplansky's question asks: is every homomorphism from $C[0,1]$ into a commutative Banach algebra continuous? Assuming $\mathrm{MA}_{\aleph_1}$ one can build a set-theoretic universe in which the answer is positive. On the other hand, in L the answer is no, so again Kaplansky's question is independent of ZFC.

Section 3: Proper forcing and the Proper Forcing Axiom

In section 1 we introduced the countable chain condition in a rather ad hoc manner, essentially to obviate the counterexamples arising in 1.4. That end might be achieved by other means. For example, regarding $\mathrm{MA}_{\aleph_1}$, the first independent instance of MA, it is natural to inquire whether there is a weak property of forcings, implied by the c.c.c., for which there is a consistent internal forcing axiom of the form: if $\mathbf{P}$ has the property, $\mathcal{D}$ is a family of dense sets and $\mathcal{D}$ has cardinality at most $\aleph_1$, then there is a $\mathcal{D}$-generic filter G in $\mathbf{P}$. How should one look for such a property? Well, in this context, the important point about MA and $\mathrm{MA}_{\aleph_1}$ is the relative consistency theorem 1.10. One could start by analysing the proof of 1.10. This is one of the tasks in Shelah's monograph [18, p.200]. We review briefly the ideas to motivate the concept of proper forcing and the Proper Forcing Axiom PFA.

The basic strategy in the relative consistency proof of MA + ¬CH is to start from a set-theoretic universe V_0 (in which CH holds) and to build a bigger set-theoretic universe $V_\star$ in which MA + ¬CH holds. We build $V_\star$ in stages and each stage is called an iteration. The construction of a stage goes roughly as follows. Given a set-theoretic universe V, a forcing $\mathbf{P} \in V$ and a filter G in $\mathbf{P}$ which is generic over V (i.e. G is $\mathcal{D}$-generic where $\mathcal{D}$ is the set $\{D \in V : D \text{ is dense in } \mathbf{P}\}$), then there is a smallest set-theoretic universe $V[G]$ such that $V \subseteq V[G]$ and $G \in V[G]$. Except in trivial cases, $G \notin V$, so $V[G]$ is a bigger universe than V. For example, if $\mathbf{P}$ is the Cohen forcing of 1.1 and G is generic over V, then the Cohen real $\bigcup G$ is a real belonging to $V[G]$ but not to V. Now extending V to $V[G]$ is not without potential danger. For example, suppose that $\aleph_1^V$ is the first uncountable cardinal in V; if $V[G]$ should chance to contain a function from $\mathbf{N}$ onto $\aleph_1^V$, then $\aleph_1^V$ is a countable set in $V[G]$, so that $\aleph_1^{V[G]}$, the first uncountable cardinal in $V[G]$, is greater than $\aleph_1^V$. In this situation, we say that $\mathbf{P}$ *collapses* $\aleph_1$. If on the other hand $\aleph_1^V$ is $\aleph_1^{V[G]}$, then we say that $\mathbf{P}$ *preserves* $\aleph_1$. The proofs that $\mathrm{MA}_{\aleph_1}$ and MA + ¬CH are consistent rely on three principal facts: (1) If $\mathbf{P}$ is c.c.c., then $\mathbf{P}$ preserves $\aleph_1$; (2) there is an iterative operation under which the class of c.c.c. forcings is closed; (3) MA_κ is equivalent to MA_κ^-. (We actually verified (3) in 1.11.)

From this very brief sketch we learn that each property of forcings for which analogues of facts (1), (2) and (3) obtain, will give rise to a consistent internal forcing axiom. One of the most interesting and powerful among these properties is properness. There are several equivalent definitions of properness. We give one which allows an easy proof that proper forcings preserve $\aleph_1$.

Definition 3.1. (1) Let A be an uncountable set. We use $[A]^{\aleph_0}$ to denote the collection of countable subsets of A. A subset C of $[A]^{\aleph_0}$ is a *club* (closed unbounded set) iff (i) every element of $[A]^{\aleph_0}$ is contained in an element of C and (ii) for every increasing sequence $x_0 \subseteq x_1 \subseteq \dots \subseteq x_n \subseteq \dots$, $x_n \in C$, the union $\bigcup_{n\in\mathbf{N}} x_n \in C$.

(2) A subset S of $[A]^{\aleph_0}$ is *stationary* in $[A]^{\aleph_0}$ iff $S \cap C \neq \emptyset$ for every club C.

(3) For a set-theoretic universe M and a set $A \in M$, we write $([A]^{\aleph_0})^M$ for the set $\{x \in M : \text{in } M,\ x \text{ is a countable subset of } A\}$.

Definition 3.2. A forcing $\mathbf{P} \in V$ is *proper* iff for every uncountable set $A \in V$, if $S \in V$ is stationary in $([A]^{\aleph_0})^V$, then S is stationary in $([A]^{\aleph_0})^{V[G]}$ for every filter G in $\mathbf{P}$ generic over V. Loosely put, proper forcings preserve stationarity.

To exercise these definitions a little, let's prove proper forcings preserve $\aleph_1$.

Theorem 3.3. *Suppose that $\mathbf{P} \in V$ is proper and G is a generic filter over V. If in $V[G]$ the set a is a countable set of ordinals, then in V there is a countable set b of ordinals such that $a \subseteq b$. Thus $\aleph_1^V = \aleph_1^{V[G]}$.*

Proof: Since in $V[G]$ a is countable, there is an uncountable cardinal λ with $a \in ([\lambda]^{\aleph_0})^{V[G]}$. In $V[G]$,

$$C = \left\{x \in ([\lambda]^{\aleph_0})^{V[G]} : a \subseteq x\right\}$$

is a club. But $S = ([\lambda]^{\aleph_0})^V$ is stationary in $([\lambda]^{\aleph_0})^V$, hence S is stationary in $([\lambda]^{\aleph_0})^{V[G]}$ since $\mathbf{P}$ is proper. Therefore $S \cap C \neq \emptyset$. Choose $b \in S \cap C$.

Definition 3.4. The *Proper Forcing Axiom* PFA is the hypothesis: if $\mathbf{P}$ is a proper forcing, $\mathcal{D}$ is a family of dense sets in $\mathbf{P}$ and $\mathcal{D}$ has cardinality at most $\aleph_1$, then there is a $\mathcal{D}$-generic filter G in $\mathbf{P}$.

The Proper Forcing Axiom is the analogue of $\mathrm{MA}_{\aleph_1}$ for proper forcings. We finish by noting some basic theorems.

Theorem 3.5. (1) *If $\mathbf{P}$ is a c.c.c. forcing, then $\mathbf{P}$ is proper. So* PFA *implies* $\mathrm{MA}_{\aleph_1}$. (2) PFA *implies* $\mathrm{MA}_{\aleph_2}$ *is false.*
(3) PFA *implies* $2^{\aleph_0} = \aleph_2$. *So* PFA *implies* MA.

Theorem 3.6. *If* ZFC+ *"there is a supercompact cardinal" is consistent, then* ZFC + PFA *is consistent.*

The large cardinal axiom in 3.6 is used to establish the appropriate version of 1.11 for proper forcings. There are variants of PFA which do not require any large cardinal axioms for their consistency proofs. There are also even stronger axioms (Martin's Maximum MM) which are studied in the literature.

Section 4: Bibliographical notes

Martin's Axiom is the eponymous subject of the monograph [6, p.200]. Good brief introductions to MA are [19, p.200], [17, p.200], chapter 2 in [11, p.200] and perhaps [22, p.200].

On Q-sets, see [14, p.200]. NMSC is covered in [20, p.200] and [5, p.200]. The articles [7, p.200] and [16, p.200] provide good accounts of the impact of logic and recent set theory. The book [4, p.200] is an excellent text on set-theoretic methods in algebra, with many applications of MA and PFA. The lecture notes in [3, p.200] deal with MA in analysis (Kaplansky's conjecture); [15, p.200] presents the solution to the Alexandroff problem and is an introduction to non-metrizable manifolds.

Proper forcings and variants appear in [18, p.200]. Applications are in [1, p.200], [2, p.200], [9, p.200] and [4, p.200]. A very interesting variant of PFA which does not require a large cardinal axiom in its consistency proof can be found in [13, p.200].

An extensive account of large cardinal axioms is provided in [10, p.200] or in [8, p.200]. [8, p.200] and [11, p.200] cover all the axiomatic set theory which we didn't. Iterations are treated in [1, p.200], [11, p.200] and [9, p.200]. [21, p.200] has the proof of 3.5 (3).

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EXPLICIT RELATIONSHIPS BETWEEN ROUTH-HURWITZ AND SCHUR-COHN TYPES OF STABILITY

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Abstract: Given two linear systems of differential equations with real or complex coefficients, and of the same arbitrary dimension. Suppose both systems are stable, one in the Routh-Hurwitz sense and the other in the Schur-Cohn sense. We directly express the coefficients of each system in terms of those of the other. These relationships, being explicit, make it possible to convey any stability criterion of either of the two types to the other.

1. Introduction

The concept of stability in differential equations has been defined in many different ways. Among these various definitions are the well-known Routh-Hurwitz and Schur-Cohn types of stability. Given a linear system of differential equations, the classical Routh-Hurwitz problem is that of obtaining necessary and sufficient conditions for all eigenvalues of the system to lie in the left half of the complex plane. The Schur-Cohn problem is that of establishing necessary and sufficient conditions for all eigenvalues to lie within the unit circle. Solutions to these problems have been the subject of intensive research over the last few years [2], [3], [9], [12] and [14].

It is often noticed in the literature that some interesting results about stability, in the Hurwitz sense for example, triggers an

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interest in the corresponding problem in the Schur sense or vice versa. See for example the introduction in [11] and example 4.2 in [8].

Recently some notable attempts have been made to give a common interpretation to the algorithms for testing the stability of continuous-time (Routh-Hurwitz) and discrete-time (Schur-Cohn) systems of differential equations [6], [10] and [13]. An excellent survey is given in [1] for continuous-time and in [5] for discrete-time systems.

The search for a unified approach to the study of root distribution of complex polynomials with respect to the half plane for continuous systems, and with respect to the unit disc for discrete systems, has been advocated by many eminent researchers in the field, see for example [4]. An interesting way of looking at the two problems of stability is to relate them to each other through the bilinear transformation $z = (1+w)/(1-w)$, which is equivalent to $w = (z-1)/(z+1)$. This is a one-to-one mapping between the left half of the complex z-plane, i.e. the region $\Re(z) < 0$, and the unit disc $|w| < 1$ in the complex w-plane. For a general discussion of bilinear transformations in this context, see [7]. Such connections prove useful in gaining new insights into the nature of the different algorithms.

This paper is a further thrust towards a firm unified approach to the relevant testing procedures for both continuous-time and discrete-time systems. In section 2 we give some notations, and the main results of the paper are given in section 3.

2. Notations

If A is an $n \times n$ real or complex matrix, and $X(t)$ is an n-dimensional column vector function of t, let $X' = AX$ be a system of differential equations, with eigenvalues $z_1, z_2, \ldots, z_n$. Then the characteristic polynomial of this system may be written in both factored and expanded forms as follows: $f(z) = \prod_{j=1}^{n}(z-z_j) = \sum_{j=0}^{n} a_j z^{n-j}$ where $a_0 = 1$ by definition. Similarly if $X' = BX$ is a system with eigenvalues $w_1, w_2, \ldots, w_n$ (where w_j is related to z_j of the previous system by $w_j = (z_j-1)/(z_j+1)$), then its characteristic polynomial is $g(w) = \prod_{j=1}^{n}(w-w_j) = \sum_{j=0}^{n} b_j w^{n-j}$, with $b_0 = 1$.

The intimate relationship between Routh-Hurwitz and Schur-Cohn types of stability could best be expressed by the following:

Theorem 2.1. *The system $X' = AX$ is Schur-Cohn stable if and only if $X' = BX$ is Routh-Hurwitz stable.*

Proof: Suppose $z = \dfrac{1+w}{1-w}$, or equivalently $w = \dfrac{z-1}{z+1}$, where z and w are complex numbers. The following relationships can easily be established

$$w + \overline{w} = \frac{2(z\overline{z}-1)}{|z+1|^2} \quad \text{and} \quad z\overline{z} - 1 = \frac{2(w+\overline{w})}{|1-w|^2},$$

from either of which it follows that $|z| < 1$ if and only if $\Re w < 0$.

3. Main Results

If $X' = AX$ and $X' = BX$ are the two systems defined in section 2 with their corresponding characteristic polynomials, then

Theorem 3.1.

$$b_p = \frac{\displaystyle\sum_{t=0}^{n} \sum_{q=\max(t-p,0)}^{\min(n-p,1)} (-1)^q \binom{t}{q} \binom{n-t}{n-p-q} a_t}{\displaystyle\sum_{t=0}^{n} (-1)^t a_t}$$

for all $p = 1, \ldots, n$.

Proof: Consider $f(z) = \sum_{t=0}^{n} a_t z^{n-t}$ with zeros $z_1, \ldots, z_n$ and $z_j = (1+w_j)/(1-w_j)$ for $j = 1, \ldots, n$. Hence $w_1, \ldots, w_n$ are the zeros of

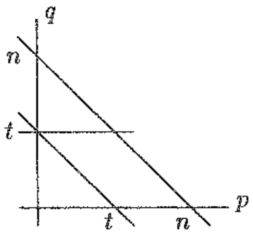
$$f\left(\frac{1+w}{1-w}\right) = \sum_{t=0}^{n} a_t \left(\frac{1+w}{1-w}\right)^{n-t}$$
$$= \frac{1}{(1-w)^n} \sum_{t=0}^{n} a_t (1-w)^t (1+w)^{n-t}.$$

Therefore $w_1, \dots, w_n$ are the zeros of the polynomial

$$\begin{aligned} h(w) &= \sum_{t=0}^{n} a_t(1-w)^t(1+w)^{n-t} \\ &= \sum_{t=0}^{n} a_t \sum_{r=0}^{t}(-1)^r\binom{t}{r}w^r\sum_{s=0}^{n-t}\binom{n-t}{s}w^s \\ &= \sum_{t=0}^{n}\sum_{r=0}^{t}\sum_{s=0}^{n-t}(-1)^r\binom{t}{r}\binom{n-t}{s}a_t w^{r+s}. \end{aligned}$$

We make the following transformation from the (r,s) plane to the (p,q) plane:

$$p = n - r - s, \quad q = r.$$



Then the rectangle in the (r,s) plane with sides $r=0$, $r=t$, $s=0$, $s=n-t$ is transformed into the parallelogram in the (p,q) plane with sides $q=0$, $q=t$, $q=n-p$, $q=t-p$. Hence

$$h(w) = \sum_{t=0}^{n}\sum_{p=0}^{n}\sum_{q=\max(t-p,0)}^{\min(n-p,t)}(-1)^q\binom{t}{q}\binom{n-t}{n-p-q}a_t w^{n-p}.$$

Write $h(w) = \sum_{p=0}^{n} N_p w^{n-p}$, where

$$N_p = \sum_{t=0}^{n}\sum_{q=\max(t-p,0)}^{\min(n-p,t)}(-1)^q\binom{t}{q}\binom{n-t}{n-p-q}a_t.$$

In the polynomial $h(w)$, the leading coefficient is

$$N_0 = \sum_{t=0}^{n}(-1)^t\binom{t}{t}\binom{n-t}{n-t}a_t = \sum_{t=0}^{n}(-1)^t a_t.$$

Now the two polynomials

$$\frac{1}{N_0}h(w) = \sum_{p=0}^{n}\frac{N_p}{N_0}w^{n-p} \quad \text{and} \quad g(w) = \sum_{p=0}^{n} b_p w^{n-p}$$

being both monic and having the same set of zeros are identical, leading automatically to the desired conclusion.

The converse of Theorem 3.1 states the following

Theorem 3.2.

$$a_p = \frac{\displaystyle\sum_{t=0}^{n}\sum_{q=\max(t-p,0)}^{\min(n-p,t)}(-1)^{p+q-1}\binom{t}{q}\binom{n-t}{n-p-q}b_t}{\displaystyle\sum_{t=0}^{n} b_t}$$

for all $p = 1, \dots, n$.

The proof of this theorem is omitted as it is similar to that of Theorem 3.1.

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UNDERGRADUATE PROJECTS

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Abstract: The rationale for, operation of and assessment of undergraduate projects at the University of Ulster are discussed. Specimen project titles are provided.

Introduction

A debate on undergraduate mathematics teaching in Ireland has recently been started through this Bulletin [1], [2]. It has been continued at a conference organized by the Sub-Commission for Mathematical Instruction of the Royal Irish Academy and held in Dublin in September 1991 (RIA-91), [3].

O'Reilly [1] questioned how we teach mathematics at tertiary level, leaving readers with many "focusing questions" and "questions for exploration". Dickenson et al. [2] described innovative methods of teaching, learning and assessment used at the University of Ulster, and went some way to answering O'Reilly's questions. Ted Hurley (UCG) continued the debate in his plenary lecture "Mathematics at Third Level" at RIA-91. In his lecture he pointed out that

(i) the number of honours graduates in mathematics from Irish Universities per capita is 3.5 times smaller than the number per capita from British Universities;
(ii) 45% of these Irish graduates entered further study compared to 14.5% of British graduates.

He concluded that this was an unsatisfactory state of affairs and made some suggestions for remedying the situation such as putting greater emphasis on the links between mathematics and computing.

The ensuing panel discussion at the Conference [3], on "Developments in Teaching Mathematics" produced other suggestions for making courses more attractive and more relevant to the needs of students seeking employment as mathematicians outside academia. One suggestion from Anthony O'Farrell (Maynooth) was to incorporate a project as part of undergraduate courses.

Project work has been a compulsory part of undergraduate mathematics at the University of Ulster, and before that at the Ulster Polytechnic, since the institution was opened in 1971. At the Ulster Polytechnic, Mathematics was offered as one (or two) subjects within a Combined Sciences degree and a final year project was undertaken in one of the main subjects studied. The University of Ulster has honours and ordinary degrees in Mathematics, Statistics and Computing [2], and includes a final year project. This article is an attempt to share with colleagues our experiences in operating projects and to provide some ideas for use or for further discussion.

Rationale for Project Work

A project is an extensive piece of work carried out individually by students under supervision. They write and submit a full report on their work and the assessment is based largely on this report. At the University of Ulster, the project comprises one module out of six taken by students in their final year. It contributes equally with other units to the final year assessment and grading of students and they are expected to spend about one-sixth of their time working at it. We have been able to arrange that one day per week is entirely free of lectures and this provides an opportunity for students to work at their project.

The aims of the project module, as specified in the Course Handbook are

(i) to involve the student in a substantial task whose completion within a strictly limited time interval will require inventiveness, knowledge of the subject areas of the course, the ability to exercise critical evaluation and judgements, and the ability to work to tight deadlines and properly to pre-plan schedules of work,

(ii) to simulate a situation in professional practice in which the mathematician must be capable of bringing all appropriate resources to bear on the solution of a particular problem,

(iii) to develop the student's ability to think independently, work without tight supervision, and make soundly based decisions,

(iv) to encourage initiative and creativity,

(v) to allow study in depth of a topic which either is not fully treated in the lectures or involves the integration of diverse subject areas,

(vi) to expose the student to a situation in which familiar techniques have to be applied in relatively complex and perhaps unfamiliar settings,

(vii) to stress the importance of a literature search, making use of modern techniques of information retrieval,

(viii) to give further experience in written and oral communication and in the production of a coherent and lucid technical report.

These give an indication of the extent of the project and of the skills which students should develop by undertaking it. In addition, students are expected to make appropriate use of computing hardware and software, and almost all projects require a substantial amount of computing because this is an essential skill for today's mathematicians. The student treats this as a research project and learns many of the basic skills of research work. While it would be useful to give the student an original problem to work on, this is not always possible and indeed not altogether necessary since students will have to make the problem "their own" and contribute their own thinking and doing to various aspects of it. We include lists of all topics we have used later in the article.

Organization

The process for the organization of final year projects begins in the second term of the penultimate year. Staff in the Mathematics Department are asked to submit titles to the project co-ordinator. It is recommended that each member of staff submit at least two titles and preferably more, with the guarantee that not more than two will be allocated to students. In addition, some colleagues from other departments are happy to assist by suggesting a title for projects which they would be prepared to supervise. Where the title is insufficient to make it clear to the project co-ordinator what the project will involve, a short description is requested.

When the complete list of titles has been compiled, it is sent to the students, who are away on placement. They are asked to reply by a fixed date, listing a few project titles in order of preference. It is also suggested that, as an alternative, they provide their own project as a result of their experiences in placement. Such a project must not be duplication of the work done in placement, which is separately assessed anyway, but may be an extension of it.

It is not surprising that some project titles are more popular than others and may be the first choices of several students. However, with 16 staff and about 25 students, it has been found possible to allocate projects without having to go below the third preference, and even then it has been necessary to go as far down as the third usually only when a student has replied after the specified date and other allocations have been made.

It is considered to be essential that each student should have a supervisor who is a member of the Mathematics Department so that there is someone who is ultimately responsible for ensuring that the project proceeds satisfactorily. Thus, if the project chosen is not one of those offered by a member of the Department, someone within it is asked to act as joint supervisor. In asking colleagues to act in that capacity consideration is given by the project co-ordinator as to whether a particular mathematical skill is required as well as to the overall supervisory responsibilities of colleagues in an attempt to share the duties fairly evenly. Once student and supervisor(s) have been matched up, it is left to both parties to make contact.

It was recommended that work should start a few weeks before the beginning of the academic year. To a considerable extent the arrangements for students meeting supervisors during the lifetime of the project is left to those concerned. While it is important that regular meetings are held so that the momentum of the work is sustained, their frequency and duration depend on factors such

as the amount of assistance required and the supervisor's personal methods of carrying out their duties. A short interim report was requested from the students by the end of January. That gave all parties some idea of what had been done, what remained to be done and whether any remedial action needed to be taken. The report does not influence the final assessment. The main report had to be submitted by the end of the first week of the Easter vacation. However, students were strongly recommended to have their reports ready by the end of the second term; the extra week was available only to take care of last minute hitches in printing and binding. In the future, University regulations will require that the time-scale be changed. Instead of there being six modules each lasting for a year, there will be three modules in each of two semesters. Thus the project will have to be completed in about half the number of weeks formerly available, although the total number of hours should remain about the same. The effects of this change on the nature and management of projects will be of interest.

Assessment

The importance placed on the project has been mentioned above. In the case of the first cohort of the degree in Mathematics, Statistics and Computing each project report was read by two assessors and marks were awarded on the basis of the report and on the work done during the year. Where a project was supervised jointly the assessors were the two supervisors; in the case of single supervision another member of the Department is the second assessor. Such a person need not have any particular expertise in the subject area of the project, since students are required to present their reports in such a way that they can be readily understood by anyone who has reached the same general level as themselves in the taught units of the course. Initially the assessors mark the projects independently and justify their conclusions in written reports. They then discuss their findings and agree a common mark.

Marks are awarded according to the following scheme.

Presentation, organization and Clarity	35%
Content and Results	30%
Student Understanding	20%
Student Effort and Initiative	15%

It can be seen from the first heading that much emphasis is placed on the student's ability to communicate through the written word; this is one of the main ways in which the project is different from the taught units. Clearly, it is not the case that both assessors are able to award marks under the fourth heading. If an assessor were involved only slightly or not at all in the supervision, he or she must rely on the other's judgement in awarding a mark under that heading. As with the taught units the assessment of the project is subject to the review of the external examiner, and the award of marks is subject to their approval.

The procedure for the assessment of the first cohort of the degree in Mathematics, Statistics and Computing was much the same as it was in the degree in Combined Sciences in which the Department participated. In the latter, only a small number of students did projects in Mathematics each year and the Department was not involved in the assessment of projects in other areas. As a result it was impossible to determine whether the projects were being assessed according to a uniform standard. However, there were 22 projects in the first cohort of the degree in Mathematics, Statistics and Computing and all members of the Department were assessors. It was then possible to look more carefully at the problem of uniformity. It has been suggested that, after the assessors have reported, two people should take an overview of all the projects, and, in consultation with the assessors, modify the marks so that they reflect the rank order. Clearly, such a scheme would create extra work but would lead to greater confidence in the uniformity of assessment.

Specimen Titles

1. Computer-aided analysis and design of circular waveguides.
2. Planetary rings.

3. Student selection and predicting student success: a Bayesian analysis.
4. Topographic mapping — a study of algorithms.
5. The application of sensitivity analysis in linear regression to detect potential carriers of Duchenne muscular dystrophy.
6. The development of a computer graphics package for portfolio diversification of securities using quadratic programming.
7. Attitudes to mathematics.
8. The problem of multiple visits in clinical trials.
9. A statistical measurement of plagiocephaly in babies.
10. A generalized printer.
11. War games and arms races.
12. A numerical study of diffusion-reaction equations.
13. PERT network CAD package.
14. Investigation of hidden line removal within drawing programs.
15. A computer-based octree modelling system.
16. To ascertain the economic advantage of notifying the retrospective testing procedure for defective output.
17. Development of a software package for the administration of placement.
18. A study of population models using differential equations.
19. The complex eigensolution of symmetric and unsymmetric matrices.
20. Development of a rule-based expert system giving guidance to architects and builders on the housing needs of disabled people.
21. Examination of relationship between meteorological and environmental factors and the growth of fungi in an area of mixed forest.
22. Conway's theory of games.
23. Structure of AA' for design matrices.
24. Modelling the duration of spells in a geriatric hospital.
25. Mathematica: a system for doing mathematics.
26. The simulation of a CNC routing cell using Witness.
27. Identifying the mathematics potential of students entering access courses.

28. Trajectories of supersonic projectiles subject to realistic air-resistance.
29. An LL(1)-grammar conformance checker.
30. Testing for quantitative and qualitative interactions in clinical trials.
31. Preparation of open learning material via Symbolator and Latex.
32. An introduction to the analytic hierarchy process.
33. Design of a computer package for a central heating system.
34. Optimum coupling distribution in waveguide design.
35. The prisoner's dilemma and similar games.
36. Design of a computer package for assisting in the drawing up of class and staff timetables.
37. The modelling and analysis of cranial evoked potentials.
38. Computerization of a credit union.
39. Numerical approximation of zeros of polynomials using methods of complex analysis.
40. Implementation and investigation of a technique for digital power spectrum estimation: DASE.
41. Stock Controller.
42. Development of an expert system.
43. The use of the median and range of a uniform distribution as indicators of quality characteristics in process analysis.
44. Simulation of a dynamically nested load for a computer numerical control (CNC) routing machine.
45. Analysing clinical trial end-points using pre-treatment information.
46. Generalized procrustes analysis.
47. The effects of an observation on the determination of a regression equation.
48. The effect of temperature on the sales of solid fuel.

Outline of two Project Tasks

1. Numerical Approximation of Zeros of Polynomials using Methods of Complex Analysis.

Results from complex analysis can be used to locate the zeros of polynomials. Numerical Algorithms based on these results have been designed. The project involves

(i) understanding the basis of the algorithms, first published in the 1950's;
(ii) writing programs to carry out approximate zero-finding;
(iii) investigating techniques for determining a whole set of zeros of a polynomial.

2. Mathematica: a System for doing Mathematics

This program will perform symbolic manipulation, including differentiation and integration, carry out numerical methods, work with complex functions, evaluate most common special functions and display results on screen or printer. The project explores some aspects of this.

Conclusion

In this article we have outlined the rationale for project work at undergraduate level and described the operation and assessment of the scheme. We have given examples of project titles recently and currently undertaken by students. We hope colleagues will find this helpful if they introduce project work to their own degree courses.

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Book Review

A BASIC COURSE IN ALGEBRAIC TOPOLOGY

Graduate Texts in Mathematics 127

William S. Massey

Springer-Verlag, 1991, 428 pages
Price DM108, ISBN 3-540-97430-X

Reviewed by Graham Ellis

In the minds of many people, algebraic topology is a subject which is "esoteric, specialized, and disjoint from the overall sweep of mathematical thought." This straightforward introduction to the subject, by a recognized authority, aims to dispel that point of view by emphasizing: (i) the geometric motivation for the various concepts and: (ii) the applications to other areas. The book, which is stripped of all unnecessary technicalities, would be a nice text for a one-year MSc course for students with a basic knowledge of point-set topology and group theory. It consists of updated material from the first 5 chapters of the author's earlier book *Algebraic Topology: An Introduction* (GTM 56) together with an updated version of almost all of his book *Singular Homology Theory* (GTM 70).

Chapter I is a 31-page partial account of the classification of compact surfaces. The following 5 chapters contain a thorough introduction to the fundamental group and covering spaces. Chapter VI explains how problems in 19th century analysis motivated the development of homology theory. (There is an appendix at the end of the book, intended for readers with a knowledge of differentiable manifolds, in which De Rham's theorem is proved.) The remaining 9 chapters are devoted to singular homology theory and cohomology theory. In order to simplify proofs Massey

has chosen to develop these theories cubically rather than simplicially. Chapter XIV contains the Poincaré duality theorems for manifolds, as well as the Alexander duality theorem and the Lefschetz-Poincaré duality theorem for manifolds with boundary. Chapter XV treats the Hopf invariant of a map from a $(2n-1)$-sphere onto an n-sphere.

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Book Review

HEAT KERNELS AND DIRAC OPERATORS

Grundlehren der mathematischen Wissenschaften

N. Berline, E. Getzler and M.Vergne

Springer-Verlag, 1992
ISBN 3-540-53340-0 (Berlin) or 0-387-53340-0 (New York)

Reviewed by Brian Dolan

This book provides a modern unified approach to index theorems for elliptic operators on compact manifolds. The book grew out of a seminar in 1985 at MIT and would be useful both for researchers with some prior knowledge of differential geometry, wishing to deepen their understanding, as well as for workers in the field. It is mathematical in its approach, as befits its subject, and assumes familiarity with such topics as fibre bundles, connections and cohomology theory. Despite devoting the first fifty or so pages to general background on differential geometry I don't think it would be a good place for someone with no previous knowledge of the subject to learn the fundamentals. The aim seems to be to set up the notation rather than explain the concepts, and the notation rapidly becomes quite involved. The book is carefully laid out in logical sequence and a little careful study is well rewarded. Typographical errors are rare, but do exist. The index is rather short, but seems adequate. There are one hundred references to some of the most important publications in the subject and the authors freely admit that this is by no means exhaustive. Only six of the references predate 1960 which gives some indication of of the historical development of the subject and it is amusing to note that, despite the title, there is no mention of Dirac's 1928 paper! There is an extensive list of notations at the back of the book which I

found extremely useful, in fact indispensable, in trying to find my way through the symbols. An indication of the style of the book is given by the fact that the index is four and a half pages long while the list of notations is three and a half pages long.

After the first chapter a general framework is developed, in terms of generalized Dirac operators on vector bundles with a $\mathbf{Z}_2$ grading, giving rise to the concept of a *supertrace* on the space of fibre endomorphisms and a *superconnection* on the vector bundle which is a first order differential operator odd under the $\mathbf{Z}_2$ grading. The definition of a Dirac operator that the authors adopt is general enough to encompass all the usual first order operators. A key ingredient of their construction is the one to one mapping between the space of exterior forms on a differentiable manifold M and the Clifford algebra for a vector space with a metric, when the vector space is viewed as a fibre of the tangent bundle of M, which they refer to as the *quantization* map.

The index theorem of Atiyah and Singer is proven, using the heat kernel approach, and its application to the four classic complexes is exhibited, giving proofs of the Gauss-Bonnet theorem, the Hirzebruch signature theorem, the index theorem for the Dirac operator and the Riemann-Roch-Hirzebruch theorem. The authors then go on to treat the equivariant index theorem, which generalizes the Atiyah-Singer index theorem to the case where there is a group action on the manifold M which is compatible with (i.e. commutes with) the generalized Dirac operator. Thus the kernel of the Dirac operator forms a representation space for the group. The equivariant index is then a generalization of the character of a group element in a given representation, its supertrace, and the equivariant index theorem relates this to an integral over the fixed point set of the group action, which is a subset of M for non-trivial group actions. Along the way the authors dispose of the Atiyah-Bott fixed point formula, where the fixed points consist of isolated, non-degenerate points. They then go on to state and prove a version of the equivariant index theorem which holds when the group element is near the identity, which they term the Kirillov formula, by analogy with Kirillov's formulas for the characters of Lie groups.

Lastly the index bundle and Bismut's index theorem are considered. The *index bundle* is defined for a family of Dirac operators by considering a fibre bundle $\pi : M \to B$ with fibres denoted by M/B. For every $z \in B$, $M_z = \pi^{-1}(z)$ is the fibre over z and $D = \{D^z | z \in B\}$ is a family of Dirac operators on M/B. If $\ker(D^z)$ has the same rank for each z the vector spaces $\ker(D^z)$ combine to form a vector bundle over B called the index bundle, $\text{ind}(D)$. The construction can be generalized to the case where the rank of $\ker(D^z)$ depends on z. Bismut's index theorem then relates the character of a superconnection for the family D to an integral over M/B. It is of interest to physicists as it has proved to be useful in string theory and the theory of moduli spaces of Yang-Mills fields.

In addition there are general chapters on equivariant differential forms and the exponential map, relating the $\hat{A}$ genus to the Jacobian of the exponential map of the Lie algebra of $SO(n)$, as well as a section on zeta functions.

In summary I found the book stimulating and rewarding, as it brought me a little more up to date in a subject which I know a little about but am not an expert in, but a thorough reading and understanding would require a larger investment of time than I can presently afford.

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Book Review

REPRESENTATION THEORY. A FIRST COURSE

William Fulton and Joe Harris

Springer–Verlag, New York, Berlin, Heidelberg, 1991. xv+551 pp.
ISBN 0-387-97495-4.

Reviewed by Rod Gow

Let me say right at the beginning that I think the authors Fulton and Harris have produced an excellent book, a book which displays a novel approach to its subject matter and is genuinely informative. Too often one feels that a textbook is largely a recitation of known techniques and ideas, with little evidence that the author has tried to find worthwhile examples or new approaches to difficult problems. However, this book presents a substantial amount of unfamiliar material in a way that is pleasing to a mathematician who has a reasonable knowledge of modern concepts of algebra.

The main subject matter of the book under review is the description of the irreducible complex representations of the simple Lie algebras over **C**. While we have a description of these representations in terms of highest weight modules, due to É. Cartan and Weyl, the emphasis in the book is directed towards explicit realization of the representations wherever possible, using the methods of multilinear algebra, symmetric polynomial theory and invariant theory. More detail is expended on the classical Lie algebras, which fall into four infinite families, since these are more accessible as algebras of vector space endomorphisms and their representations may be studied rather more explicitly than those of exceptional algebras such as F_4, E_6, E_7 and E_8. A worthwhile feature of the authors' approach is the way they are able to point out interesting geometric aspects of the representations they construct. Thus, under the guise of geometric plethysm, arising in the context of the representations of $sl_2(\mathbf{C})$ and $sl_3(\mathbf{C})$, there is a discussion of the Veronese embedding of a projective space in a larger projective space and of the associated Veronese surfaces. Grassmannian varieties also occur, with a description of their Plücker embeddings into projective spaces, when the decomposition of symmetric powers of exterior powers is examined.

The authors organize the material of the book into 26 lectures, rather than chapters. There are numerous exercises throughout the book, some of which are distinctly demanding of the reader. There are, however, at the end of the book, 20 pages of hints, answers and references relating to the exercises. There are 63 pages of appendices concerning such topics as symmetric polynomials, multilinear algebra, properties of Lie algebras and classical invariant theory. There is also a bibliography of six pages. The book begins with familiar material on the complex representations of finite groups. However, more emphasis is given than usual to the problem of decomposing certain tensor products for particular groups. Indeed, already on p.31, the authors show that the exterior powers of the natural module of degree $d-1$ for the symmetric group S_d of degree d are all irreducible. While this is a known result, a head-on proof of the kind given is not usually found in the literature. Lecture 4 is devoted to the construction of the irreducible characters of S_d. This is of course a vast area of study, where numerous contributions have been made. I enjoyed the authors' presentation, which includes many interesting facts and different points of view. In Appendix A at the end of the book, there are proofs of a number of results concerning symmetric polynomials which are needed to develop the finer aspects of the character theory of S_d. In particular, Schur functions are introduced and various determinantal formulae for their evaluation are proved. The Littlewood–Richardson rule for multiplying Schur functions is described but not proved. However, a special case, known as Pieri's rule, is proved and this is often sufficient for many purposes. It is clear that much of classical determinant theory, such as one sees in Thomas Muir's *Treatise on the theory of determinants*, finds its natural home in this context. Lecture 6 presents the theory of Schur and Weyl for decomposing the n-fold tensor power $V^{\otimes n}$ of a finite dimensional complex vector space V into irreducible $GL(V)$–submodules, where $GL(V)$ is the group of all automorphisms of V. The irreducible submodules are picked out as the images of the Young symmetrizers, which are rational multiples of certain idempotents in the rational group algebra of S_n. Examples are given to show how the Littlewood–Richardson rule is used for working out, among other things, the decomposition of certain tensor products.

Part II of the book is devoted to introductory material on Lie algebras and Lie groups. While a Lie algebra is an easily defined algebraic object, a Lie group seems much more complicated, with its attendant topology and geometry. The authors make the point that the structure of the Lie algebra of a Lie group provides crucial information on the structure of the group. Moreover, the finite dimensional irreducible representations of the group may be studied via the irreducible representations of the algebra. Standard examples of real and complex linear Lie groups are given, together with less familiar examples of complex tori, related to elliptic curves and abelian varieties. It is shown how Lie algebras arise from Lie groups by taking the differential of the adjoint representation of the Lie group on the tangent space of the identity and then how to pass back to (subgroups of) the Lie group via the exponential map. A number of the basic theorems on complex Lie algebras are proved, such as those of Lie, Engel and Cartan. The Killing form is introduced somewhat later in the book. Then the irreducible representations of the most accessible of the simple Lie algebras over $\mathbf{C}$, $sl_2(\mathbf{C})$, are investigated. This material, while straightforward, is vital for understanding the structure of arbitrary simple Lie algebras and also plays a role in the representation theory, via the principal 3–dimensional Lie algebras that occur as subalgebras of the simple algebras. In order to describe the irreducible representations of $sl_3(\mathbf{C})$, it is necessary to develop certain ideas, such as highest weight vector and weight lattice, that play the dominant role in the representation theory of all simple Lie algebras.

Part III of the book develops some more theory relating to simple Lie algebras, special cases of which were encountered when investigating $sl_2(\mathbf{C})$ and $sl_3(\mathbf{C})$. The irreducible representations of the special linear, symplectic and orthogonal Lie algebras are constructed and investigated from a variety of points of view, which I found most instructive and helpful. In order to understand the missing representations of the orthogonal Lie algebras, which cannot be constructed from tensor operations on the natural module, the authors introduce the Clifford algebra and spin groups. This material again is handled in a very clear conceptual manner. There is also a brief discussion of the triality automorphism of order 3 in the Lie algebra $so_8(\mathbf{C})$. Part IV of the book contains a variety of material, which we can scarcely do justice to. Among more familiar topics are Weyl's character formula, the weight multiplicity formulae of Freudenthal and Kostant and Cartan's classification of real simple Lie algebras. Less familiar topics discussed include the connection between g_2 and skew–symmetric trilinear forms defined on a seven–dimensional vector space and algebraic constructions, due to Freudenthal, of the exceptional Lie algebras, where, again, trilinear forms play a basic role. The relationship between the Clifford algebra based on an eight–dimensional vector space, octonions (Cayley numbers), g_2 and the triality automorphism is also briefly explained (g_2 is, as far as I know, the fixed algebra of the triality automorphism).

There are a number of good introductory books on Lie algebras and their representations, such as those Jacobson and Humphreys. Bourbaki, in his three volumes on Lie theory, provides a large amount of material as well, much unavailable elsewhere, but the reader is often asked to find proofs from minimal hints. I think that this book, with its concentration on examples to illustrate and interpret the theory, is a most useful addition to the literature in this area. Its style will appeal to pure mathematicians but people, such as physicists, who occasionally have recourse to Lie theory, should also find something worthwhile for them when studying this text. I did not notice very many obvious typographical errors in the book. The authors seem to have prepared the typescript themselves by computer and this may account for the good quality of the work. The

authors cannot decide on the spelling of octonion, which appears variously as octonion and octonian, sometimes on the same page. The name of the Italian mathematician Trudi is given as Trudy and Gordan, of Clebsch–Gordan fame, has his name consistently misspelt as Gordon. The word principle appears for principal on p.422 and there is a $\wedge$ sign missing from formula B7 on p.474. However, the overall impression created in this reviewer was that of an ambitious text, skilfully worked and interspersed with novel observations, which any library or researcher, experienced or novice, might purchase without regret.

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