ARTICLES

Gödelian Incompleteness and Paraconsistent Logics

Or: why Gödel's Paradox is really a dilemma

D.H. Cohen

There are three parts to this largely expository discussion. First, Gödel's Incompleteness Theorem is differentiated from other sorts of questions in mathematics. From this a classification scheme for some of mathematics' more perplexing situations emerges[1]. The Incompleteness result is then presented, explained, and located in that scheme. Finally, in light of this, and some points of language and philosophy made in passing, a new response to Gödel's result is suggested.

1 Puzzles and other problems

Gödel's Theorem poses an altogether new *sort* of problem for mathematicians. It establishes that some very good mathematical questions do not have equally good mathematical answers. Moreover, the fault is not in what counts as a mathematically acceptable question; it is in what counts as a mathematically acceptable answer. The problem is endemic to the entire mathematical enterprise.

The idea that some good questions do not have equally good answers is troubling but hardly new. There are many other situations involving answerless questions without anyone suggesting that they represent crises for human thought. But Gödel's proof *does* pose just that sort of crisis. Identifying just what distinguishes it from similar problems is the best first step towards

framing a rational response. To that end, comparisons with other notable open questions in mathematics will be made. The first part of the strategy is to uncover the implicit theoretical framework supporting judgements that some problems *are* similar to others. The results are then applied to Gödel's Theorem in particular.

Consider "Fermat's Last Theorem." It might be thought similar to Gödel's Theorem since it may be an example of an undecidable question[2]. A good many subcases have been closed, but no comprehensive proof or disproof has yet been found. Only such (dis-)proof would qualify as a definitive answer. As a working hypothesis, however, a good assumption is that some proof or refuting counterexample will be produced eventually. Without this assumption, further progress could not reasonably be expected. It is considered a problem with no solution *yet*, and problems with no solutions yet are simply unfinished business. They are like the intellectual itch of a particularly difficult daily crossword puzzle the day before the solution is printed. Since the same community response and attitude is present, the term **PUZZLE** is appropriate. The assumption is that there is a unique solution[3].

The same general response seems to be called for with respect to the humanly uncheckable proof of the Four-Colour Theorem. Although it introduces an empirical component into the practice of pure mathematics, that practice can be sufficiently distanced from theory so as not to pose any conceptual problems. The theorem is either true or false; the proof is either valid or not. Human capacities are not the issue. Any faith in Platonic Realism is not really challenged.

A second sort of unanswered question does raise problems for an uncritical realism, for the belief that there is a truth of the matter out there for us to discover or not, depending on the acuteness of our powers and the blessings of the mathematical muse. Cantor's Continuum Hypothesis typifies this set. Is the cardinality of the continuum the second transfinite cardinal? What differentiates this problem from Fermat's is that we know the resources at hand are *insufficient* to decide it. The hypothesis is provably *consistent* with but *independent* of the axioms of set-theory: either it or its negation can be added to those axioms without yielding contradiction ...provided only that the original axioms were consistent. The situation is analogous to Euclid's "fifth postulate". Who knows but a range of "Non-Cantorian" set-theories are just waiting to be developed — waiting, presumably, in the same place that Non-Euclidean geometries were waiting for Riemann and Lobaschevsky, wherever that was[4].

Gödel's own position was an extreme Platonism, maintaining that the Continuum Hypothesis was either objectively true or false, depending on whether or not it accurately described the behaviour of transfinite cardinals[5]. They are out there, just like Kurdistan or Pago Pago, and it is the business of the mathematician to be a geographer of this special realm of abstract entities. Even the idea that there are alternative set-theories waiting to be discovered is a kind of Realism, albeit pluralistic about the theories themselves.

This sort of situation is a DILEMMA. Several options are available, any one of which can be developed and applied. Thus, the basis for choosing one rather than another must be based on something *external* to the different systems, say, the way the world is in its ultimate metaphysical construction, or the desiderata surrounding some specific research program or computational context. The situation would be analogous to a jigsaw puzzle that could be put together in two different ways or a single crossword grid and set of clues that could be "correctly" solved in several distinct ways[6]. In a dillema, the existing conceptual framework needs additional information. What is accepted or established may be fine as far as it goes, but it simply doesn't go far enough.

The opposite situation also arises, cases where we have, as it were, too much "information". In this case, something has to be discarded to resolve the issue rather than something having to be added to settle things. The search, then, is for a likely belief to jettison. This is a PARADOX. The common pattern is that established and accepted theses give rise to an absurdity, or even an outright contradiction. Russell's Paradox is an example of this: the set of all sets that are not members of themselves must be — but cannot be — a member of itself. This amounts to a contradiction; something has to go. We could, if we were so inclined, abandon the belief that the world is contradiction-free, or that sentences cannot be both true and false simultaneously, or that set-theory is worth pursuing. The consensus has been that the unrestricted version of the set-abstraction axiom, although "obvious", is the source of the problem. Obviousness is not always the mark of truth. Restricting its range of applicability preserves just about all of naive set-theory. Nothing so radical as an overhaul of the underlying logic is required. So, by an implicit appeal to a principle of "minimal mutilation," restricting the relevant axiom is the change that is usually made[7].

Russell's Paradox, while forcing some revision in mathematical beliefs and practice, presents no real threat to the main Platonist tenet of objective mathematical truth. It merely challenges the secondary assumption that some particular statement of the set-abstraction axiom is part of that truth. A mistake was made, big deal. Of course, at the time of its discovery it was not as easy to be so blithe about revision. We know now what Russell did not know then: effectively restricting the schema is entirely possible and not unduly burdensome.

Ironically, this sort of Realism, which is Gödel's own position, is rather less viable when confronted by Gödel's Incompleteness Theorem. The theorem presents a conceptual paradox requiring revision of very fundamental beliefs, and possibly even the logic that holds everything together.

2 Gödel's Theorem

The theorem establishes, in brief, that there is an unavoidable mismatch between mathematical truths and mathematical theorems. The two sets cannot coincide. No matter how arithmetic is packaged, there must be either some truths that elude the proof-theoretic apparatus, or else some falsehoods that sneak their way into theoremhood.

Considerations of space prevent a complete rehearsal of the details of Gödel's proof here. However, an approximation is at hand, something a bit easier, but still in the same neighbourhood. It can be proved (perhaps contrary to expectations but provable nonetheless) that no mathematician is omniscient. By "omniscient" I mean believing *all and only* the true sentences. If it were simply a matter of believing everything that is true, it would be relatively easy: believe everything. Believe that $2 + 3$ is 5 but also believe that $2 + 3$ is not 5, that it is 6 and that it is not, that cabbages are kings and that they are not, that taxes can be lowered and government revenues raised at the same time, and so on. Similarly, believing *only* truths is also relatively easy: don't believe anything. Doubt that $2 + 3$ is 6 and doubt that it is not, and so on. (This may not be such a bad idea. Descartes tried it and managed to get his co-ordinates straight.) The trick, clearly, is to manage both at the same time, to believe all the truths *and* disbelieve all the falsehoods. Now consider this sentence:

(N) Prof. N. Üllset does not believe this sentence

(letting Prof. Üllset represent an arbitrary mathematician). If he believes it, it is false and he has a false belief. If he does not, it is true and there is a truth not in his belief-set. Either way, sadly, he falls short of omniscience.

This is what Gödel managed to do for arithmetic. He showed that any language rich enough to express what arithmetic needs to express will also be able to express an equivalent of N-sentence, saying roughly, "Arithmetic cannot prove this sentence".

The proof centers on the notion of *computability*. The idea is that all functions of a certain type ought to be expressible in any language suitable for mathematics. The type in mind, *recursive functions*, is conservatively characterized: a few (rather boring) functions are taken as primitive and means for constructing new ones are provided. The given functions are (1) the constant zero-function, (2) the successor function, and (3) projection functions which simply pick out the i^{th} member of a given n-tuple. Additional functions may be built up either by (4) function-composition or by (5) recursion. That's it.

The next stage involves showing that such purely syntatic concepts as *term, function*, and *well-formed formula* (wff) are representable by constructible functions. The vehicle for this is an assignment of numbers to each concatenation of symbols in the language. It is then shown that the *sets of numbers* corresponding to terms, wffs, and the rest, can be defined by recursive functions. For example, a one-place recursive function can be constructed which has the value 1 if and only if its argument is a number corresponding to a well-formed formula; otherwise, it has the value 0.

In addition to concepts that are syntactic in the grammatical sense, some concepts which are syntactic in the proof-theoretic sense are similarly representable by recursive functions. These include the concepts of *axiom, substitution instance*, and even of *proof*! That is, whether or not a given sequence of formulas is a valid proof is the kind of question that can be answered by a Türing machine: a program can be written which will correcting answer, after a finite number of steps, "yes" or "no" to the question "Is this collection of symbols a legitimate proof?"

Theoremhood, however, is *not* recursive. Given a sequence of wffs, it can be definitely decided that it does or does not constitute a proof. If it does, then the last wff is certainly a theorem. What cannot be devised is an effective test which starts with a single formula and correctly answers yes, it *can* be the last line of a proof sequence, or not it *cannot.*

This means that arithmetic is *undecidable.* There is no alogirithm for theoremhood. Undecidability leaves it open as to whether there might be proofs that are undiscover*ed* or even recursively unrecover*able* for every mathematical truth. The proofs could be out there, alongside the undeveloped non-Cantorian set-theories, waiting for discovery or doing whatever it is that unknown truths do. However, Gödel's Theorem actually proves something stronger than undecidability, so this picture is wrong. Gödel's Theorem is an *incompleteness* theorem, proving that at least some of those imagined proofs for each and every truth are not there — no matter where "there" is.

The way this works is that *theoremhood* is an expressible concept within the language of recursive functions, although not itself recursive. The notion of proof is recursive, so theoremhood is easily recoverable. If $ded(x,y)$ holds just in case x is the number associated with a sequence of formulas that is a proof and y is the number associated with the formula which is the last line of that proof, then $Th(y)$, defined as $(Ex)ded(x,y)$, defines theoremhood. The predicate Th holds of just those numbers associated with theorems.

The hard part is re-creating the kind of self-reference that is the N-sentence — "Prof. N. Üllset does not believe *this* sentence". The recursiveness of substitution allows that. Let $sub(x,n,a)$ represent the substitution relation. Or, more exactly, the value of the $sub(x,n,a)$ is the number associated with the formula that results from substituting the expression associated with the number n for the variable associated with the number x in the formula associated with the number a.

Now, consider the formula:

$$\mathbf{I}: \quad \neg Th(sub(k_x, x, x)),$$

where k_x is the number associated with the symbol for the variable x. This entire formula is itself associated with some number, its "Gödel number". Let it be i. Now consider the formula:

$$\mathbf{J}: \quad \neg Th(sub(k_x, i, i)).$$

This says that the result of substituting the number i for the variable x in the formula with Gödel number i is not a theorem. The formula with Gödel number i is I, so this says that the result of substituting i for x in I is a non-theorem. The result of that substition is precisely J, so J says in effect, "J is not a theorem". If it is a theorem, it's a false theorem, if it is not a theorem, it is a true non-theorem. Mathematics, no matter how axiomatized, is not "omniscient". (The notion of proof is system relative, so expanding the system by adding the unprovable sentence as a new axiom wouldn't help; a new "Gödel sentence could always be generated.)[8]

Exactly why theoremhood is expressible but not recursive concerns the quantifier in its definition: $(Ex)ded(x,y)$ — *there is* some number answering to a sequence of formulas which is a proof of the given formula. There is no upper bound that can be given on the length or complexity of the proof simply from the syntactic complexity of the putative theorem. We could, were we so determined or demented, ask if 1 is the number of a proof of a given wff y, then ask if perhaps 2 is, then 3, and so on. Practical considerations aside, this method would find a proof of y, if there were one, eventually. But if there were *no* such proof, it would continue indefinitely, never getting a negative answer. There is no point at which one could say, "There has been no proof yet so none exists".

The situation is analogous to a variation of Goldbach's Conjecture. Let us call an even number a "Goldbach Number" if it is indeed the sum of two prime numbers. For any given even number n it can be determined definitively whether or not it is a Goldbach number. Simply check all the pairs of natural numbers whose sum if n. There are only finitely many pairs to check. Suppose, in contrast, we wanted to know whether n is the *difference* of two primes instead of the sum, a "Bach-gold Number" instead of a Goldbach Number. The quantification in this version of the conjecture is unbounded — "there are two primes such that ..., " not, as was implicit in the first case, "there are two primes *less than n* ..." — so there are infinitely pairs to test.

As with Bach-gold numbers, no limit can be established beforehand on how high up the ladder of natural numbers one has to climb before one can confidently assert that some formula is *not* a theorem.

3 Gödel's Dilemma.

The impossibility of an "omniscient" axiomatization of arithmetic is no less than that of an omniscient mathematician. But just as a mathematician is given two choices — either a false belief or an unbelieved truth — so too arithmetic has two choices: *inconsistency* or *incompleteness*. That is, if we are willing to consider inconsistency as a viable possibility, then the absurdity of mathematical incompleteness is no longer paradoxical; it is more of a *dilemma*!

But is inconsistency a viable option? Classically, no. Standard truth-functional accounts of implication maintain that from a single contradiction anything whatsoever may be legitimately inferred. A set of wffs G has the wff A as a logical consequence if it is impossible for all of the members of G to be true while A is false. When G is inconsistent, i.e., when its members cannot be simultaneously true, trivially satisfied for every A. The proof-theory mirrors this semantic account. A simple proof of B from $A\&\neg A$ is:

1.	$A\&\neg A$	premiss
2.	A	1, &-Elim ("Simplification")
3.	$A \vee B$	2, $\vee$-In ("Addition")
4.	$\neg A$	1, &-Elim
5.	B	3, 4 $\vee$-Elim ("Disjunctive Syllogism")

Both semantically and proof-theoretically, the admission of any inconsistency annihilates the theory.

Recent work in "Paraconsistent Logics," however, has shown that inconsistent systems can be viable. Paraconsistent logics are logics that can tolerate contradiction without degenerating into triviality. On the proof-theoretic side, this involves putting some restrictions on the patterns of proof permitted. The semantic innovation is to abandon the idea that the implication connective is entirely a truth-functional one.

Paraconsistent logics have been motivated in a variety of ways. Often, the motivation is the failure of the truth-functional analysis of if-then sentences. It fails as a model for the use of such conditionals in ordinary discourse[9] and it fails to provide the necessary conceptual framework for non-trivial reasoning from inconsistent premisses, i.e., for *reductio ad absurdum* reasoning. Moreover, if a logic is to be an information processing tool, the *pre-requisite* of consistency is self-defeating: it is doubtful how many human intellectual endeavours are consistent. And, needless to say, managing to prove consistency prior to *any* use of logic would be a great acomplishment.

The two important questions to address are the *how* and the *why* of inconsistent arithmetics. First the why: Why even consider a system that is known beforehand to have at least this one big flaw? The answer, in part, is that this might not be a flaw at all. It is important to keep in mind that part of the task of describing the world consists in *devising a language* with which to do so. Not even the most extreme Realist could deny it. Objective world or not, the choice of a vocabulary is a determinant of the shape of the resultant theory. The preliminary task of choosing or designing a language is not trivial. It is something that can be done well or poorly. It is a task using skills and criteria for success quite apart from those used in the subsequent descriptive operation. Also, there is nothing at the outset of the enterprise that guarantees success. On the contrary, Gödel's Theorem can be read as guaranteeing at least partial failure!

Success or failure aside, it is a mistake to minimize the contribution that the language makes to description. The language used makes its own appreciable contribution. Certain sentences may be certified as true *by the lanaguage itself* regardless of how the world is. "If it is raining, then it is indeed raining" is certainly true and just as certainly independent of the weather. No object can be in two places at the same time. Of course, but is this really a profound and *a priori* fact about objects or is it a perfectly natural consequence of the way we use the word "object", of the way we count objects, and of the way we decide what is to count as a single object? Even if an object could be in two places at once, we wouldn't count it as one object in two places but as two objects. Likewise, certain sentences are certified as false by the language itself. Anyone seriously asserting "It's raining but it isn't" or "Santa Clause does not exist although I sincerely believe he really does" would be guilty of a kind of linguistic incompetence.

Could the set of sentences certified as true by the language and the set certified as false by that language intersect? Nothing rules it out. This may be what the Liar's Paradox is all about. "This sentence is false" is both true and false according to the rules of language (And also neither true nor false, by those same rules)[10]. But what consequences about the world should one be able to draw from that?

This might be crucial to understanding Gödel's Theorem. Incompleteness and inconsistency represent genuine alternatives for arithmetic. Inconsistency can be accepted, if only as a pathological consequence of any language that permits self-reference. It is indeed an avenue worthy of further exploration. The underlying logic would have to be adapted accordingly, but that can be done. Localise the inconsistency; contain its effects. If it turns out that sentence J and its negation are both provable, what follows? Well, it follows that J is a theorem; it also follows that $\neg J$ is a theorem. So are both $J \vee A$ and $\neg J \vee A$ although A might or might not be deducible. There is no reason to suppose that the consequences extend to Fermat's conjecture or the next general election or why the sea is boiling hot and whether pigs have wings.

But *how* can an inconsistency be localized? Isn't the proof above incontrovertible? Two philosophers means three opinions, so of course it is not beyond dispute. Indeed, I think the proof is demonstrably fallacious — a case of "Begging the Question." Specifically, the disjunctive syllogism (DS) is the last line is illegitimate. Ordinarily, the reasoning from $A \vee B$ and $\neg A$ to B

is quite unexceptionable: if it is either A or B, and it isn't A, then it simply must be B. And if someone asks why it has to be B the answer is because it isn't A. We already know A is false. But in this particular case we also know that it *is* A! That is, if what we are given is that A is both true and false, then we cannot later appeal to the principle that nothing can be both true and false. Taking $A \& \neg A$ as a premiss requires that we suspend the principle of non-contradiction — and with it, the principle of disjunctive syllogism that relies on it.

The situation is analogous to this. Suppose I said that I think if Gauss were alive and doing his work today, he would not only be more famous than any other living mathematician, he would even be more famous than Michael Tyson, Michael Jackson, or Mikhail Gorbachev. You might respond that, sure, he's a great mathematician and would deserve it, but it's hard to believe that society would suddenly and at long last give mathematicians their due. To that I'd say, "But don't forget, if Gauss were living today, he'd be *212 years old* and how many 212 years olds are any good at mathematics at all?" The joke is obvious. Conversational implicatures demand the incorporation of certain beliefs and the suspension of others. Violating the implicit rules can have comic effects. If we are asked to suppose that Gauss were alive, we are generally meant to suppose, among other things, that this is possible and that it involves a minimal change of the result of our beliefs about the way the world is. Some beliefs must change, such as that there is no one quite like Gauss around, but most other beliefs need not be put aside, including the belief that the world just doesn't have 212 year old mathematicians in it. But it *could* have someone now pretty much like Gauss was 200 years ago, even though we may believe it does not.

The same general sort of thing is going on in the proof. If we are asked to suppose that $A \& \neg A$ then we are also asked to suppose that it is possible. That requires suspension of many other beliefs, including the universal applicability of DS. Implicitly, there is an appeal to "It can't be both A and $\neg A$" — in spite of the fact that $A \& \neg A$ is exactly what we were asked to suppose! If the information that A is true was used to get $A \vee B$, that forestalls using the information that A is *not* true, even though the negation of A is also supposed to be true. DS might or might not work for $B \vee C$ and $\neg B$, but it has been set aside for A.

Would the reformulation of arithmetic with a paraconsistent logic as its basis be workable? How much of mathematics could be recovered or reconstructed with additional restrictions on allowable methods of proof? Largely that depends on the exact restrictions adopted. Intuitionist mathematics is a relevant precedent. At the heart of Intuitionistic thought is a rejection of Platonism and an embrace of "constructivism". If mathematics is "identical with the exact part of our thought" then the process of human thought is integral to the subject. This includes, notably, the fact of human limitations and thought's temporal unfolding. The infinite is tolerated only insofar as it is exactly specified. Infinite sets are countenanced, for instance, only as potentialities and only if rules for construction can be given. Existence proofs are accepted only if they include a method for constructing that mathematical entity; a *reductio ad absurdum* of a non-existence claim would not suffice.

The Intuitionistic program has had its share of success in both logic and philosophy. For example, much work has been done on formalizing and researching Intuitionistic logics, demonstrating at least that the philosophical program can be given and exact and coherent formal basis[11]. Further, the same sorts of considerations that led to some of the negative reactions against Zermelo's original use of the axiom of choice in 1904 to prove the well-ordering theorem continue to play an important part in current debates in the philopophy of language[12]. Within mathematics proper, however, Intuitionism has had mixed results. On the one hand, the restrictions Intuitionists impose on non-constructive existence proofs undermine the whole of Cantorian transfinite arithmetic — as was desired. And much of classical mathematics can be recovered within their guidelines. On the other hand, Intuitionistic proofs can be unwieldy, and rejecting the axion of choice and its equivalents means forswearing perhaps more of set-theory and analysis than would be desired[13].

The Intuitionists' penchant for constructivism (alternatively: their squeamishness about the infinite) entails rejecting the law of the excluded middle: undecidable propositions are neither true nor false. Their logic has truth-value "gluts" or inconsistencies — sentences taking both truth- values — so is not really a paraconsistent logic. It is these logics, logics that deny the principle of non-contradiction, that are relevant here.

The best developed paraconsistent logics are from the family of systems known collectively as "Relevance Logics". Although some work has been done on Relevant arithmetics using Robinson's and Peano's axiomatisations of arithmetic and the logic systems R, RM, and E, the work has been neither as systematic nor as institutionalized as the Intuitionists'[14]. In part, this may be due to dissension in their ranks as to the appropriate system to use;[15] in part it may be simply due to the absence of a Brouwer-Heyting calibre combination of mathematician and logician. Nevertheless, there has certainly been enough success to warrant further exploration of the "Inconsistency option" with respect to Gödel's Incompleteness Theorem.

Notes

1. This taxonomy of puzzles-paradoxes-dilemmas is developed in greater detail in D.H. Cohen 1988.

2. It has been suggested (by Martin Gardner 1989, p. 26n) that if the "theorem" really is undecidable, then it must be true: if it were false, there would be a counterexample which would decide things. This assumes the system is "omega-consistent", which amounts to Realism of a sort.

3. This is similar to the use of "puzzle" in Thomas Kuhn 1970. Kuhn suggests that "puzzle-solving" is the mark of a "normal" science.

4. See. P.J. Cohen and R. Hersh 1967.

5. See Kurt Gödel 1964.

6. See D.H. Cohen 1985.

7. W.V. Quine offers an excellent general discussion of paradoxes in the title essay Quine 1966.

8. E. Nagel and J.R. Newman 1959 is an excellent introduction to the notion of incompleteness. A good, more technical account is in J.W. Robbin 1969, pp.90–119.

9. Routley 1982 contains many counterexamples and an extended polemic against the truth-functional model for conditionals. Other discussions abound.

10. See Jennings and Johnston 1983.

11. Arend Heyting's formalization of Intuitionist logic has attracted the most attention. See Haack 1974, pp.91-103. The *Journal of Philosophical Logic* special issue on Intuitionism (v. 12, no. 2, May 1983) addresses a spectrum of the logical questions.

12. M.A.E. Dummett is perhaps the leading proponent of Intuitionisti thought in logic, philosophy of mathematics, and philosophy of language Dummett 1977 offers a good introduction to Intuitionism in mathemat ics; Dummett 1978 considers it in other contexts.

13. The consistency and independence proofs for the axiom of choice hav defused the issue somewhat. See Dummett 1977 or Heyting 1972.

14. See Anderson and Belnap 1975 for a quasi-official statement of the pro gram; see Routley 1982 for dissenting voices. Volume II of *Entailment*, edited by N. Belnap and J.M. Dunn, is due out shortly. Relevantl developed systems of mathematics will be included.

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Department of Philosophy
Colby College
Waterville, ME 04901, USA