DYNAMICAL LIE ALGEBRAS

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An algebra is just a vector space with multiplication. A *Lie Algebra* has a peculiar form of multiplication referred to as commutation - concretely realized by matrices x and y

$$[x,y] = xy - yx,$$

together with the axioms following from this. There is no loss of generality in considering all finite dimensional Lie algebras as just algebras of matrices.

There seem to be at least two reasons why Lie algebras play a prominent role in modern theoretical physics:

1. *The Symmetry Aspect*: Theories which undertake to provide a description of space and time start by emphasising the underlying symmetry, before going on to give more detailed discussions of the mechanics. Thus if we say "Empty space looks the same everywhere" this is interpreted as "Translational Symmetry" (and such symmetries do in fact lead to observable conservation laws). The mathematical formulation of this type of symmetry leads to a structure called a *Lie Group*, and Lie algebras are related to Lie groups in much the same way as the additive properties of logarithms relate to the multiplicative properties of numbers.

2. *The Dynamical Aspect*: When we get down to giving a more detailed mechanical description of nature, we must provide dynamical laws, such as Newton's laws for Classical Mechanics. The latter are expressed in differential calculus form, so it is fairly obvious that the techniques of differential calculus will prove useful in a study of classical mechanics. One formulation of the basic laws of Quantum Mechanics, however, utilizes the Lie algebra commutator introduced above - for example the famous Heisenberg relation for position operator q and

momentum operator p (and unit operator I)

$$[q,p] = ihI$$

(where h is Planck's constant over 2π). If we take (q,p,I) as a basis in a (complex) vector space, then the Heisenberg relation (together with [q,I] = [p,I] = 0) gives us a Lie algebra structure (the Heisenberg algebra). It is not surprising, therefore, that Lie algebras play a pivotal role in a discussion of quantum dynamics.

It is in their second role, that of providing a dynamical description of quantum theory, that Lie algebras have become recently increasingly fashionable, as *Dynamical Lie Algebras*; this is distinct from their use in the first context, that of Symmetry Lie Algebras. To the pure mathematician, of course, there is no distinction, it is the same Lie algebra structure involved - merely the application which differs. In fact, the same (isomorphic) Lie algebra may be used in two different contexts, in one case as a symmetry algebra and the other as a dynamical algebra.

The simplest example of the preceding discussion is the algebra $so(3)$. This is defined as the set of 3x3 (real) antisymmetric matrices. As a (real) vector space, it is three-dimensional and we can choose a basis (J_1, J_2, J_3), where $[J_1, J_2] = J_3$ and the two similar commutation relations obtained by cyclic permutation also hold. This algebra is well known as the Lie algebra of the group of rotations in three-dimensional space. Rotational symmetry is, of course, an assumed symmetry of the world, and so this exemplifies the symmetry use of Lie algebraic theory. Any rotation matrix R may be written $R = \exp J$, where $J \equiv a_1 J_1 + a_2 J_2 + a_3 J_3$ is an element of $so(3)$; this makes R automatically orthogonal - and J is essentially the logarithm of R.

However, this same algebra also occurs as a dynamical algebra in, for example, the theory of superconductivity. Such quantum systems are specified by writing down a hamiltonian operator H; eigenvalues of H, for example, give the set of energy levels of the system. In the case of a simple model of superconductivity - the so-called BCS model named after the Americans J. Bardeen, L.N. Coopes and J.R. Schrieffer who developed it in 1957 - this hamiltonian may be written as

$$H = a_1\hat{J}_1 + a_2\hat{J}_2 + a_3\hat{J}_3.$$

(Here the real numbers a_1, a_2 and a_3 are related to the kinetic and potential energy of the system.) Now of course the $\hat{J}_i$ are no longer 3x3 matrices; they are operators. But they have precisely the same commutation relations as the J_i above of *so*(3). Thus H may be considered as an element of *so*(3). Note that this algebra is *not* a symmetry algebra of H; that is, H does not commute with the $\hat{J}_i$ (that is, $[H,\hat{J}_i] \neq 0$) and it does not commute with the "rotations" $\hat{R} = \exp \hat{J}$ corresponding to the $\hat{J}_i$ (that is, $\hat{R}H\hat{R}^{-1} \neq H$). However, Lie algebraic techniques can now be utilized to solve the BCS problem. We know that a 3x3 anti-symmetric matrix may be diagonalized by an orthogonal matrix. This purely algebraic result can be extended to any dimension, and indeed to the operator H above. We may thus find explicitly a "rotation" R such that, for example,

$$\hat{R}H\hat{R}^{-1} = a J_3, \ (a^2 = a_1^2 + a_2^2 + a_3^2).$$

Since J_3 may be chosen to be a diagonal operator, the spectrum of $\hat{R}H\hat{R}^{-1}$, and thus that of H, is immediate. This gives the energy levels of the system.

The most obvious use of these dynamical algebras - to obtain the spectrum of a quantum system - gives them the name Spectrum Generating Algebras. (The corresponding Lie groups are often referred to as Dynamical Groups.) Their first use in Elementary Particle Physics dates from the work of Y. Ne'eman and collaborators in the early sixties; subsequently the present writer employed the method in condensed matter physics (superfluidity, 1970) and there has recently been a resurgence in nuclear physics (the Interacting Boson Model) as well as in

straightforward potential theory. Current applications of the technique are to quantum systems exhibiting many coexisting phases simultaneously - such as superconductivity and magnetism.

I have given no references in this short, informal and introductory note; Arno Bohm and Yuval Ne'eman of Austin, Texas are currently editing a review monograph ("Spectrum Generating Algebras and Dynamical Groups", World Scientific, 1987) which will cover the history and applications of the subject.

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