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BULLETIN

The aim of the *Bulletin* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

The *Bulletin* seeks articles of mathematical interest written in an expository manner. All areas of mathematics are welcome, pure and applied, old and new.

Detailed instructions relating to the preparation of manuscripts may be found on the inside back cover.

Correspondence relating to the *Bulletin* should be sent (from March 1987) to:

Irish Mathematical Society Bulletin,
Department of Mathematics,
University College,
Galway.

EDITORIAL

As this is the last issue of the Bulletin that we shall be editing, we would like to take the opportunity of making some comments on progress.

We have been pleased to observe an increase in the number of authors contributing to the Bulletin. The variety of articles has been maintained while their quality has, if anything, improved. There is perhaps a slight oversupply of material from Cork but that should change with the move to UCG.

It is notable that until now only a tiny handful of articles has been refused completely (quite a few were accepted only after revision) and the grounds for refusal or request for revision have always been that the submitted version was too technical or would appeal to too small a minority of readers.

It would be our ambition for the future of the Bulletin that it would continue to see its primary function in publishing material which can be read with profit by a majority of members of the Society. We see no point in becoming another research journal in which the average (worldwide) readership of a given article is exceedingly small. We should set our sights high and look to the Monthly and the Intelligencer as models!

Since the format of the Bulletin was improved so dramatically in the early years of its sojourn in Cork, changes in its image are perhaps not so easy to discern. However, there has been a significant process of standardization taking place in the general layout: titles, paging, references, typescripts and so on. These improvements are largely the result of continuing efforts on the part of our typist, Leslie Brookes; our thanks go to him for his contributions.

Finally, we welcome Ray Ryan and Ted Hurley as Editor and Associate Editor respectively. We hope their experience with the Bulletin will be as pleasant as ours has been and we wish them well with their new task.

Martin Stynes   Pat Fitzpatrick

IRISH MATHEMATICAL SOCIETY

SECRETARY'S REPORT 1986

1. EUROMATH PROJECT

The appeal to the Irish third level institutions to make modest contributions towards the preliminary phase of the European Mathematical Council's proposed mathematical database was very successful. We received contributions of £50 each from the mathematics departments of the RTC in Cork, DIT (Kevin St), Maynooth, NIHE (Limerick), Thomond College, TCD, UCC, UCD and UCG, and another £50 from the UCC Statistics department - a total of £500. I understand that a proposal for major funding from the EEC's ESPRIT programme has been prepared and is now being assessed by the EEC.

2. AMERICAN MATHEMATICAL SOCIETY

We now have a reciprocity agreement with the American Mathematical Society. The agreement has been announced in the November issue of the Notices of the American Mathematical Society. It means that members of the IMS can join the AMS at a reduced rate of $42 (the full rate is $64 or $84, depending on your income). Application forms are available from me.

3. REPORT ON MATHEMATICS IN IRELAND

We are all aware that Mathematics is an essential part of the infrastructure of Science and Technology. Some of our members work in areas which are considered "applicable" but those of us who work in "pure" areas may not be as aware as we might be how close our subject is to practical applications. Also, many areas which are considered as being unquestionably "applied" have now become intertwined with very high-powered areas which were considered the core areas of pure mathematics.

There are two US reports which, I think, bring out this unity and significance of present-day mathematical research. These are the David report (reproduced in the Notices of the AMS, August 1984, pages 434-469) and a report produced recently by a committee under the chairmanship of Philip Griffiths (see Notices, AMS, June 1986, pages 462-479).

The David report has had an enormous impact on US Federal policy towards mathematics. The US government has realised, as a result of that report, that Mathematics is the cornerstone of development at the frontiers of Science and Technology. This has been translated into increased Federal funding for Mathematics research (as distinct from Computer Science research).

It has been suggested to me by a number of members that if a report were prepared here in Ireland by one of our most distinguished mathematicians with the aim of elucidating the present and potential role for Mathematics in Irish education and industry, it could form the basis for a new appreciation of the significance of Mathematics. My purpose in writing this is to start a debate on whether we should embark on such a project and how best to go about it.

Richard M. Timoney

IRISH MATHEMATICAL SOCIETY

Ordinary Meeting, December 19, 1986

An ordinary meeting was held at 12.15 pm at the Dublin Institute for Advanced Studies. The meeting began under the temporary chairmanship of Prof. S. Tobin, who handed over the chair to the Vice-President, S. Dineen. The President was unable to attend. There were 13 members present.

1. The minutes of the meeting of April 4th, 1986, were approved and signed.

2. The Treasurer presented his financial report and also reported on a substantial increase in membership from 181 to 231 since January 1986 (including reciprocity members: 22 via the IMTA and 14 via the new AMS reciprocity agreement). There was also one additional institutional member (UCD).

   F. Holland suggested that it might possibly benefit the IMS to be registered as a charity and it was agreed that this might merit investigation.

   The Treasurer's report was formally proposed and seconded and then approved unanimously.

3. The Secretary presented his report, divided into three main headings - the appeal for donations towards EUROMATH (the proposed mathematical database), the new reciprocity agreement with the AMS and a suggestion that the IMS consider preparing a report which would identify the optimal future role for mathematics in Ireland. The latter idea was discussed at length and received approval in non-specific terms. F. Holland relayed the information that EUROMATH had been approved for substantial EEC funding. The idea was raised of instigating further reciprocity

agreements (for example with the Edinburgh Mathematical Society).

4. T. Laffey reported on his and F. Holland's efforts to ensure that Ireland would be represented at the 1988 International Mathematical Olympiad, which is to be held in Australia. The Government has agreed to this and the Department of Education will provide significant funding for the trip. T. Laffey listed a number of the steps which would be necessary to select and prepare a team. The most suitable group of students would probably be those completing their penultimate year in secondary school, as the timing of the competition would essentially prohibit Leaving Certificate candidates from taking part. The next Mathematics contest is to be used to identify a significant number of suitable students (say 100 of them). This tuition would require the assistance of teachers, but it would also mean the provision of suitable books (which would have to be bought) and perhaps sets of notes on relevant topics. IMS members willing to help in this process by compiling notes or lists of problems are being sought. It might be necessary to have some "summer camps" for groups of these students so that they could receive intensive training and so that a team (maximum of 6 pupils on the team) could be selected. The team would then need an intensive training session prior to their trip.

   With a view to the future, the question of soliciting funds to send a team to the 1989 Olympiad was being considered.

   Finally, T. Laffey or F. Holland are most anxious to hear from those willing to assist in the preparation of a team.

5. The meeting agreed that the Society would not formally support the Carmen Bueno campaign, which is the successor of the now-defunct Orlov-Scharansky and Massera campaigns (which the Society did support and which achieved their objectives).

6. It was announced that R. Ryan and T. Hurley of UCG would edit the Bulletin, following the April 1987 issue.

The Secretary also announced a plan that the membership list will eventually include electronic addresses for those members who wish to supply them. In addition, it is proposed to ask each Mathematics department to set up a departmental electronic mailbox and organise that it be regularly checked (by a secretary, for example).

7. The following were proposed, seconded and elected unopposed (for two-year terms in each case):

President	:	S. Dineen	(UCD)
Vice-President	:	F. Gaines	(UCD)
Committee	:	M. Brennan	(WRIC)
		N. Buttimore	(TCD)
		B. Goldsmith	(DIT, Kevin St)
		R. Ryan	(UCG)

Richard M. Timoney

(Secretary)

LETTER TO THE EDITOR

Dept of P. & Q. Science,
Regional Technical College,
Waterford.
February 1987

Dear Editor,

In your September 1986 issue Donal Hurley and Martin Stynes reported on the basic mathematical skills of UCC students.

They concluded that students entering third-level colleges do not have "the desired basic skills". In the opinion of the authors third-level mathematics lecturers should "take an interest in and play an active role in designing the mathematics curriculum at first and second levels".

Your third-level readers ought to know that:

(1) A syllabus committee under the aegis of the Department of Education and composed of three inspectors, three teacher union representatives and three school manager representatives put the finishing touches about a year ago to three new Intermediate Certificate syllabi. The Intermediate Certificate stable door is swinging open and the horse has gone.

(2) However, a Leaving Certificate Syllabus Committee (for Mathematics) also under the Department's aegis but this time including a representative of the universities ($\neq$ third-level colleges) had its work on three new Leaving Certificate syllabi suspended, also about a year ago. The Leaving Certificate horse is still in the stable.

It is important for the authors of the UCC survey to distinguish between their viewpoint and the grounds for the long-standing objection that universities (*sic*) have been over-influencing second-level mathematics syllabi. That objection is concerned with topics such as linear programming, vector analysis, calculus, even parts of trigonometry on the Lower Leaving Course; it is concerned with linear transformations, convergence of sequences and series, probability, groups, ... on the Higher Course.

The authors of the article about UCC students want the third-level sector to influence second-level syllabi. I think I know in what way, but perhaps they would spell it out? And I would like to know what recommendations they or other third-level people would make for primary level. Finally, given the composition of syllabus committees heretofore, how are all third-level interests to be represented from now on?

Michael Brennan

AN ESSAY ON PERFECTION

Mícheál Ó Searcóid

1. ABSTRACT

A recent result of Murphy [2] concerns operators on a given infinite dimensional Hilbert space H. It states that, given a non-empty compact subset K of the complex plane, there exists a non-diagonalizable normal operator on H whose spectrum is K, if and only if K is uncountable. To effect this result, the operator theorist must use a well known result in topology, namely, that every uncountable separable metric space contains a perfect set. This excursion into topology led us to investigate how far the conditions of separability (in the metric space case this is equivalent to second countability) and metrizability can be stretched. We give below some general results about topological spaces which must contain perfect sets, and we produce a series of counterexamples to show that the conditions we have imposed on the spaces are indeed quite sharp.

2. INTRODUCTION

We recall firstly that a subset of a topological space X is said to be perfect in X if it is non-empty and is equal to the set of its limit points in X.

We now generalize the notion of compactness in the following definition: let Ω be a transfinite cardinal number and let X be a topological space. X is said to be Ω-*compact* if every open cover for X has a subcover of cardinality less than Ω. It is clear that X is compact if and only if X is $\aleph_0$-compact, and that X is Lindelof if and only if X is $\aleph_1$-compact.

In what follows, we shall be concerned also with a generalization of the concept of a G_δ. We shall consider a topological space which has the property that each singleton subset of it can be expressed as the intersection of open sets whose number is less than Ω, where Ω is a given transfinite cardinal number. We note that any space which satisfies such a property is necessarily a T_1 space. Also, when $\Omega = \aleph_1$ the property is equivalent to that of each singleton set being a G_δ. We note also that every singleton set in a first countable T_1 space is necessarily a G_δ; and further that a compact Hausdorff space has the property that each of its singleton subsets is a G_δ if and only if it is first countable.

For all other topological definitions, we follow Kelley [1].

3. TOPOLOGICAL RESULTS

We begin with an elementary lemma, which mimics the compact case:

3.1 LEMMA. *Let Ω be a transfinite cardinal number and let X be an Ω-compact topological space. Let Y be a closed subspace of X. Then Y is also Ω-compact.*

PROOF. Suppose $(U_\alpha)_{\alpha \in A}$ is an open cover for Y, where A is some indexing set. Then, for each $\alpha \in A$, $U_\alpha = V_\alpha \cap Y$ for some set V_α open in X. Then $X \setminus Y \cup (V_\alpha)_{\alpha \in A}$ is an open cover for X, so there exists a subset B of A with $\mathrm{card}(B) < \Omega$ such that $X \setminus Y \cup (V_\alpha)_{\alpha \in B}$ covers X. Then $(U_\alpha)_{\alpha \in B}$ covers Y.

We are now ready to prove our main theorem:

3.2 THEOREM. *Let T and Ω be transfinite cardinal numbers with $T \leq \Omega$. Let X be a regular topological space which is T-compact and in which every singleton subset can be expressed as the intersection of $\leq\Omega$ open sets. Suppose $\mathrm{card}(X) \geq \Omega$; then there exists an T-compact subset of X which is perfect in X.*

PROOF. Let $T = \{x \in X : \forall N \in \mathrm{Nbd}(x),\ \mathrm{card}(N) \geq \Omega\}$. We shall show that T is perfect in three stages.

(a) Firstly we show that if Y is any closed subspace of X with $\mathrm{card}(Y) \geq \Omega$, then $Y \cap T \neq \emptyset$. Indeed, suppose, on the contrary, that $Y \cap T = \emptyset$; then, for each $y \in Y$, there exists an open neighbourhood N_y of y with cardinality less than Ω. Now Y is T-compact by 3.1, so there is a subset V of Y of cardinality less than T such that

$$Y \subseteq \bigcup_{y \in V} N_y .$$

This leads to the contradictory conclusion that $\mathrm{card}(Y) < \Omega$. Hence we have $T \cap Y \neq \emptyset$; in particular, T is not empty.

(b) Secondly, T is closed in X, for, if x is any element of X and if N is a neighbourhood of x which has non-empty intersection with T, then N is a neighbourhood of some point of T and hence has cardinality not less than Ω. So T is closed in X, and T is T-compact by 3.1.

(c) Thirdly we show that every point of T is a limit point of T. Let $t \in T$, and suppose that K is a closed neighbourhood of t; then

$$\mathrm{card}(K \setminus \{t\}) \geq \Omega .$$

Let $(N_i)_{i \in I}$ be a family of open sets of X with $\mathrm{card}(I) < \Omega$ which satisfies

$$\{t\} = \bigcap_{i \in I} N_i .$$

Then

$$K \setminus \{t\} = \bigcup_{i \in I} (K \setminus N_i)$$

Hence we have $\mathrm{card}(K \setminus N_i) \geq \Omega$ for some $i \in I$.

Part (a) of the proof now allows us to deduce that

$$T \cap (K \setminus \{t\}) \neq \emptyset .$$

Since X is a regular space, this is sufficient to conclude that t is a limit point of T.

Hence T is perfect in X.

Some notes regarding the separation property of our topological space are in order. Firstly, a T_1 space is regular and compact if and only if it is compact Hausdorff. Secondly, a regular Lindelof space is necessarily normal. These considerations account for the formulation of the following two special cases of our theorem:

3.3 <u>COROLLARY (i)</u>. *Let X be a first countable compact Hausdorff space. If X is uncountable, then X contains a perfect set.*

(ii) *Let X be a normal Lindelof space in which every singleton set is a G_δ. If X is uncountable, then X contains a perfect set.*

<u>PROOF</u>. This is precisely what theorem 3.2 says when

(i) $\tau = \aleph_0$ and $\Omega = \aleph_1$.

(ii) $\tau = \Omega = \aleph_1$.

It is well known that a compact Hausdorff space is metrizable if and only if it is second countable. It is well to remind the reader at this point that there do exist first countable compact Hausdorff spaces which are not second countable. For an example, consider $[0,1] \times [0,1]$ in the order topology induced by dictionary order: $(a,b) < (c,d)$ means that either $a < c$ or both $a = c$ and $b < d$.

Actually, since every second countable space is hereditarily Lindelof, the general theorem for these spaces is much more easily stated:

3.4 <u>THEOREM</u>. *Let X be an uncountable second countable topological space. Then X contains a perfect set.*

<u>PROOF</u>. Let $T = \{x \in X: \forall N \in Nbd(x), N \text{ is uncountable}\}$. Since every subspace of X is Lindelof, an argument similar to that of 3.2(a) shows that T has non-empty intersection with every uncountable subset of X. So $X \setminus T$ is countable, and it follows immediately that every point of T is a limit point of T. That T is closed is proved as in 3.2(b). Hence T is perfect and the result is proven.

We should like to investigate now the necessity or otherwise of the toplogical conditions which we placed on the space X in theorem 3.2 in order to ensure a successful outcome. The following three examples are instructive:

3.5 <u>EXAMPLE</u>. Let X be any set with the discrete topology. Then X is Hausdorff since all subsets are clopen; X is first countable since each singleton subset is open; and X is locally compact since each singleton set is clopen and compact. Yet, whatever its cardinality, X contains no perfect set because all its points are isolated.

3.6 <u>EXAMPLE</u>. Let Y be an infinite set and let $y \in Y$. We define a toplogy on Y by declaring as open each set whose complement is finite or whose complement contains y. Then Y is Hausdorff since there is only one singleton set in Y which is not clopen; Y is compact since each open cover for Y contains a neighbourhood of y, which necessarily has finite complement; yet, however large the set Y is, Y contains no perfect set, since only one of its points is not isolated.

3.7 <u>EXAMPLE</u>. Let Z be any non-empty set and let $z \in Z$. We define a topology on Z by declaring as open each subset of Z which does not contain z, and Z itself. Then Z is compact since every open cover for Z contains the only neighbourhood of z, namely Z; Z is first countable since z has exactly one neighbourhood and every singleton set other than $\{z\}$ is open.

Z is a T_0 space for the same reason. Yet Z, regardless of its cardinality, contains no perfect set since only one of its points is not isolated.

This last example is not as satisfying as the other two. Although Z is T_0, it is not T_1, and although it is first countable, not every singleton set can be expressed as the intersection of open sets. Ideally we should have liked to find an uncountable compact first countable T_1 space which contains no perfect set.

Example 3.5 shows us that the essential role played by T-compactness in the proof of 3.2 cannot be assumed by any local property. It is true, however, that a local property is sufficient to provide us with a converse of the most special case of our theorem:

3.8 <u>THEOREM</u>. *Let X be a locally compact Hausdorff space. If X contains a perfect set P, then every set in the relative topology of P, other than ∅, is uncountable.*

<u>PROOF</u>. Suppose P is a perfect set in X, and let U be an open set in X which has non-empty intersection with P. Then $\overline{U \cap P}$ is closed in X so is locally compact. Now, $\overline{U \cap P}$ is a T_1 space so each of the sets $(U \cap P) \setminus \{t\}$ $(t \in U \cap P)$ is open in $\overline{U \cap P}$. Furthermore, since P is perfect, each of these sets is also dense in $\overline{U \cap P}$. Now, $\overline{U \cap P}$ is a locally compact regular space so that Baire's theorem holds; hence $\bigcap_{t \in I} (U \cap P) \setminus \{t\}$ is dense in $\overline{U \cap P}$ for any countable subset I of $U \cap P$. It follows that $U \cap P$ is uncountable.

We are now in a position to state a necessary and sufficient condition for a certain type of topological space to contain a perfect set. Moreover, we can identify that part of the space in which perfect sets must lie:

3.9 <u>THEOREM</u>. *Let X be a first countable compact Hausdorff space. Then X contains a perfect set if and only if X is uncountable. In that case, every perfect set in X is contained in T, where T = {x ∈ X: every neighbourhood of x is uncountable}.*

<u>PROOF</u>. By 3.2 and 3.8.

Of course it is not true that all perfect sets are uncountable. The most primitive counterexample to that conjecture is an indiscrete space of two elements. This is compact, first countable and perfect in itself. A more formidable counterexample would be any countably infinite set with the finite complement topology. This has all the above properties and is a T_1 space besides.

We have proved a converse to 3.3(i). Any attempt to produce a converse in general to the main theorem is, however, doomed to failure. In fact, the converse to our second special case is easily seen to be false. The set of rational numbers with the usual metric gives us a normal first countable T_1 space, as all metric spaces do, which is Lindelof since it is countable. This set is clearly perfect in itself, yet is not uncountable.

This last counterexample brings to prominence that perpetual defect of the rational numbers - that they are incomplete. Complete metric spaces behave well in the present context. Indeed, theorem 3.8 has a companion theorem, proved in exactly the same way:

3.10 <u>THEOREM</u>. *Let X be a complete metric space. If P is a perfect set in X, then every set in the relative topology of P, except for ∅, is uncountable.*

It should be added that complete metric spaces do not necessarily contain perfect sets even when they have large enough cardinality. Indeed, any set can be endowed with the discrete metric to produce a complete metric space; as we have already noted, no discrete space contains a perfect set.

In metric spaces, we can look for perfect sets which are small in the sense of the metric. Our main theorem yields us a result:

3.11 THEOREM. *Let Ω be a transfinite cardinal number. Let X be an Ω-compact metric space and suppose* card(X) $\geq \Omega$. *Let ε be a positive real number; then there exists an Ω-compact subset of X which is perfect in X and whose diameter is not greater than ε.*

PROOF. The open balls $\{x \in X: \text{dist}(x,a) < \varepsilon\}$ $(a \in X)$ cover X; therefore some subset of them of cardinality less than Ω also covers X. Since card(X) $\geq \Omega$, it follows that at least one of the balls, say B, has cardinality not less than Ω. Now $\bar{B}$ is Ω-compact by 3.1; being a metric space, $\bar{B}$ is also T_3 and first countable. Theorem 3.2 now furnishes us with an Ω-compact perfect set in $\bar{B}$, which is also of course Ω-compact and perfect in X. Its diameter does not exceed ε.

This leads us to a very special case indeed, where we can say a little more:

3.12 THEOREM. *Let X be a subspace of $\mathbb{R}^n$, where n is a natural number. Let ε be a positive real number. We have:*

(a) If X is uncountable, then X contains a perfect set of diameter not more than ε; further, if X is closed, then this perfect set is both perfect in $\mathbb{R}^n$ and compact in $\mathbb{R}^n$.

(b) If X contains a perfect set, then the closure $\bar{X}$ of X in $\mathbb{R}^n$ is uncountable.

PROOF. (a) Since every subspace of $\mathbb{R}^n$ is Lindelof, the first part is given by 3.11. The resulting perfect set P in X certainly has no isolated point in $\mathbb{R}^n$ and, if X is closed, it is closed in $\mathbb{R}^n$, hence perfect in $\mathbb{R}^n$. Since P is bounded it is also compact by the Heine-Borel theorem.

(b) If P is perfect in X, then it is clear that $\bar{P}$ is perfect in $\bar{X}$. Since $\bar{X}$ is locally compact, the result follows from 3.8.

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Roinn na Matamaitice,
Colaiste na hOllscoile,
Corcaigh,
Eire.

EXTENSIONS AND K-THEORY OF C*-ALGEBRAS

G.J. Murphy

INTRODUCTION

The theory of C*-algebras is increasingly having an impact on other areas of Mathematics, and on Mathematical Physics, for example, on Algebraic Topology, Differential Geometry, Topological Group Theory and Quantum Mechanics. Our aim here is to give an account, comprehensible to the non-specialist, of some of the most important recent results in this subject.

THE BROWN-DOUGLAS-FILLMORE THEORY

Let H be a Hilbert space (all vector spaces and algebras are over the complex number field C). An operator T on H is *normal* if $T^*T = TT^*$ and such an operator is *diagonalizable* if H admits an orthonormal basis consisting of eigenvectors of T. Of course relative to such a basis T has diagonal matrix, and on finite dimensional Hilbert spaces all normal operators are diagonalizable, but this is false in infinite dimensions: if $(e_n)_{n \in Z}$ is an orthonormal basis for H and if T in B(H) (the algebra of all bounded linear operators on H) is defined by $Te_n = e_{n+1}$ $(n \in Z)$ then T is normal and a trivial calculation shows that T has no eigenvectors.

However despite this negative result, in a certain sense normal operators are "nearly diagonalizable". To be precise, H. Weyl (1909) showed that if H is a *Hermitian* operator ($T = T^*$) on a separable infinite-dimensional Hilbert space H then T is a sum of a diagonalizable operator and a compact operator (an operator K on H is *compact* if there is a sequence of operators K_n with finite dimensional ranges such that $||K_n - K||$ converges to 0 as n tends to ∞, where $||.||$ denotes the operator norm on B(H). From the point of view of Operator Theory compact operators are "small", and adding on a compact operator to a given operator only "perturbs" the operator "inessentially").

By the way, the Weyl result fails if separability is dropped. Surprisingly, the extension of this result to all normal operators did not come until 1970 when I.D. Berg showed that every normal operator on a separable infinite dimensional Hilbert space is the sum of a diagonalizable and a compact operator.

Now let us consider the set $\mathcal{D}+\mathcal{K}$ of all sums D+K where D is a diagonalizable and K a compact operator on H (*henceforth H will always denote a separable infinite-dimensional Hilbert space*). Given an operator T on H one could ask for a "spectral condition" on T that T belong to $\mathcal{D}+\mathcal{K}$. This is not very precise, but if $T \in \mathcal{D}+\mathcal{K}$ then its self-commutator $T^*T - TT^*$ is compact, i.e. T is *essentially normal*. One could now ask (naively) do all essentially normal operators belong to $\mathcal{D}+\mathcal{K}$? The answer is no, and the explanation is elementary but revealing.

An operator S on H is *Fredholm* if it has closed range and the spaces N(S) and N(S*) are finite dimensional (N( ) denotes the null-space or kernel). We then define the *Fredholm index* of S to be

$$\text{index}(S) = \text{dimension } N(S) - \text{dimension } N(S^*).$$

One has index(S) = index(S+K) for all compact operators K, and if S is normal, index(S) = 0. Now let $(e_n)_{n \in N}$ be an orthonormal basis for H and let U be the operator on H defined by $Ue_n = e_{n+1}$ $(n \in N)$. U is called the *unilateral shift* and will be referred to again later. U is essentially normal and of Fredholm index -1. Thus U cannot be of the form diagonal + compact, since any such operator is of index 0.

It turned out that this index obstruction was the *only* obstruction, but the proof of this required the introduction of homological algebra techniques into Operator Theory. First we shall state the results of the beautiful theory of L. Brown, R. Douglas and P. Fillmore (1973) and then we shall indicate briefly their approach to the problem.

If $S \in B(H)$ its *essential spectrum* is the set $\sigma_e(S)$ of all complex numbers λ such that $S-\lambda 1_H$ is not a Fredholm operator.)

THEOREM (B-D-F, 1973)

1. *If T is an essentially normal operator on H then T is a sum of a diagonalizable operator and a compact operator if and only if* index$(T-\lambda 1_H) = 0$ *for all* $\lambda \in C \setminus \sigma_e(T)$.

2. *If* T_1, T_2 *are essentially normal operators on H then ther is a compact operator K on H such that* T_2-K *is unitarily equiv alent to* T_1 *(in this case we say* T_1 *and* T_2 *are compalent if an only if* T_1 *and* T_2 *have the same essential spectrum X and for all* $\lambda \in C \setminus X$ *we have* index$(T_1-\lambda 1_H)$).

Thus this theorem completely classifies the essentially normal operators (a very large and important class of operator up to unitary equivalence modulo the compact operators. Although the results are stated in simple Operator Theoretic terms the proofs involve algebras of operators, i.e. C*-algebras, an as mentioned above, homological algebra. Many Operator Theorists would prefer proofs that did not involve the latter, and seems that at last this may be possible, for only this year (1986) I.D. Berg and K. Davidson have announced a new proof o the B-D-F theorem that apparently uses quite different method However, homological methods are here to stay in Operator Alg ebra Theory, since there now exist many more deep results usi these methods, some of which we'll be looking at later.

THE THEORY OF EXTENSIONS

A *C*-algebra* is a Banach algebra A with an isometric involution $x \mapsto x^*$ such that $||x^*x|| = ||x||^2$ for all x in A. I X is a compact Hausdorff space then C(X), the set of all comp valued continuous functions on X, is a C*-algebra with the ob ious pointwise-defined operations and the supremum norm. If is any Hilbert space, then B(K) is a C*-algebra (the norm is the operator norm, and the involution is defined by the

usual adjoint operation). All self-adjoint closed subalgebras of B(K) are C*-algebras and the Gelfand-Naimark theorem says that every C*-algebra has a faithful representation as such a C*-algebra.

If A and I are C*-algebras then an *extension* of A by I is a short exact sequence of C*-algebras and *-homomorphisms

$$0 \to I \to E \to A \to 0.$$

(If A,B are C*-algebras a **-homomorphism* from A to B is an algebra homomorphism $\alpha: A \to B$ which preserves the involution, $\alpha(x^*) = (\alpha(x))^*$ for all x in A. We say α is *unital* if A and B have multiplicative identity elements 1_A and 1_B and $\alpha(1_A) = 1_B$.)

Our definition of extension is too general for the present purpose, since we shall only be interested in extensions of C(X), for X a compact Hausdorff space, by K(H), the C*-algebra of all compact operators on H. Thinking of extensions as short exact sequences is a little clumsy, so we shall present them in an equivalent but more convenient form.

Henceforth X denotes a compact metrizable space.

An *extension* of C(X) (by K(H)) will mean an injective unital *-homomorphism $\tau: C(X) \to B(H) / K(H)$ (this quotient algebra is a C*-algebra with the quotient norm and obvious involution: it is called the *Calkin* algebra). We say two extensions τ_1, τ_2 of C(X) are *equivalent* if there exists a unitary operator U in B(H) (i.e. $U^*U = UU^* = 1$) such that $\tau_2(f) = \pi(U)\tau_1(f)\pi(U^*)$ for all f in C(X). Here π denotes the quotient map from B(H) to B(H) / K(H). This defines an equivalence relation and we denote the class of τ by $[\tau]$, and the set of these equivalence classes by Ext(X). We'll see shortly that Ext(X) can be made into a group.

Now let $T \in B(H)$ be essentially normal. Then $\pi(T)$ is a normal element of the Calkin algebra, i.e. $\pi(T)$ and $\pi(T)^*$ commute, so by the Spectral Theorem there exists a unique unital injective *-homomorphism τ_T from $C(\sigma_e(T))$ to the Calkin algebra such that $\tau_T(z) = \pi(T)$, where z denotes the inclusion map of $\sigma_e(T)$ in C. We call τ_T the *extension of $\sigma_e(T)$ determined by* T. If two essentially normal operators on H both have essential spectrum X then they are compalent iff the extensions they determine are equivalent.

Given a general compact metrizable space, extensions of C(X) exist. In fact trivial extensions exist, where the extension τ is said to be *trivial* if there is a unital *-homomorphism ρ from C(X) to B(H) such that $\tau = \pi\rho$. The trivial extensions form the zero of Ext(X). (By the way if metrizablity of X is dropped then trivial extensions may not exist.)

The first important result of this theory is that all trivial extensions of C(X) are equivalent. The proof uses Weyl's theorem, and Berg's theorem drops out as a consequence of this result. An addition can be defined on Ext(X) in a natural way (using direct sums of operators), and one can show easily that Ext(x) is a commutative semigroup. The fact that the class of the trivial extensions forms the zero of Ext(X) is a non-trivial result - using it and the Wold-von Neumann decomposition of isometries (an *isometry* is an operator U in B(H) such that $U^*U = 1$) one can show the following:

<u>THEOREM (B-D-F, 1973)</u>. *Let $U \in B(H)$ be the unilateral shift and let $T \in B(H)$ be an essentially unitary operator (i.e. $\pi(T)^*$ is the inverse of $\pi(T)$ in the Calkin algebra, so in particular T is essentially normal), and let n be the Fredholm index of T. Then there exists $K \in B(H)$ compact such that*

1. *T-K is unitary if n = 0.*
2. *$T-K = U^n$ if n is negative.*
3. *$T-K = U^{*n}$ if n is positive.*

It follows that Ext(T) = Z where T is the unit circle.

As mentioned earlier Ext(X) is a group. The original B-D-F proof of this was very complicated but W. Arveson has simplified the proof.

Our next task is to "identify" the group Ext(X), at least for X a compact subset of C (the case relevant to single operators).

Let $\pi^1(X)$ denote the first cohomotopy group of X (this can be identified as the quotient group of the group of invertible elements of C(X) modulo the connected component of 1 (which is a subgroup). Equivalently $\pi^1(X)$ is the group of homotopy classes of continuous functions from X to $C \setminus \{0\}$. $Hom(\pi^1(X), Z)$ denotes the group of all homomorphisms from $\pi^1(X)$ to Z, and we define a map γ_X from Ext(X) to this group by the equation

$$\gamma_X[\tau][f] = \text{index}(\tau(f))$$

where $[\tau] \in Ext(X)$, f is an invertible element of C(X), and [f] denotes the class of f in $\pi^1(X)$. $\gamma = \gamma_X$ is easily seen to be a homomorphism, but it is not an isomorphism is general. However it is a deep result of the B-D-F theory that γ is an isomorphism if X is a compact subset of the plane. It is here that homological algebra comes in, and surprisingly perhaps, one has to be able to talk about Ext(X) for X not a subset of the plane to construct the proof.

We are now ready to sketch a proof of the B-D-F theorem: let T_1, T_2 be essentially normal operators on H with essential spectrum X and suppose that $\text{index}(T_1-\lambda 1) = \text{index}(T_2-\lambda 1)$ for all $\lambda \in C \setminus X$. We have to show that the extensions τ_1 and τ_2 determined by T_1 and T_2 respectively are equivalent extensions, and to do this it suffices to show that $\gamma_X[\tau_1] = \gamma_X[\tau_2]$. But $\pi^1(X)$ is generated by the elements $[z-\lambda_\omega]$ where z is the inclusion map of X in C and λ_ω is an arbitrary point of the hole ω (a *hole* of X is a bounded connected component of $C \setminus X$). Thus it suffices to show that $\gamma_X[\tau_1][z-\lambda_\omega] = \gamma_X[\tau_2][z-\lambda_\omega]$, i.e.

$\text{index}(\tau_1(z-\lambda_\omega)) = \text{index}(\tau_2(z-\lambda_\omega))$, i.e. $\text{index}(T_1-\lambda_\omega) = \text{index}(T_2-\lambda_\omega)$. But this is true since $\lambda_\omega \in C \setminus X$.

Brown, Douglas and Fillmore went on to show that one can use Ext to define a generalized periodic homology theory on compact metric spaces. Their theory thus links up with the theory of pseudodifferential operators and the Atiyah-Singer index theorem. We are not going to pursue this; instead we look at a theory which is in a sense dual to that of extensions, which has already had many important applications, and which is in some respects more "natural" than the theory of extensions.

K-THEORY OF C*-ALGEBRAS

The basic idea of K-theory is that we can analyse a C*-algebra in terms of the projections and unitaries that it - or rather the matrix algebras $M_n(A)$ - contain. To avoid technical difficulties we assume that A is unital. A *projection* in A is a self-adjoint idempotent element $p : p = p^2 = p^*$. A *unitary* u in A is an element whose adjoint is its inverse. We let $M_\infty(A)$ denote the set of infinite matrices with entries in A and with only finitely many entries non-zero. With the obvious matrix operations this is an involutive normed algebra (each subalgebra $M_n(A)$ has a unique norm making it a C*-algebra, and $M_\infty(A)$ is the union of all $M_n(A)$ $(n = 1,2, \ldots)$).

We say projections e,f in A are *equivalent* if there is a continuous path of projections in $M_\infty(A)$ from e to f. H(A) denotes the set of equivalence classes [e] for this equivalence relation. If $s,t \in H(A)$ then there exist projections e,f in $m_\infty(A)$ such that $ef = 0$ and $s = [e]$, $t = [f]$. We define (without ambiguity) $s+t = [e+f]$. This makes H(A) an abelian semigroup with zero element, and we let $K_0(A)$ denote its Grothendieck enveloping group (loosely speaking the set of all formal differences $[e] - [f]$). The definition of $K_0(A)$ for A non-unital is got from the unital case. The details

are unenlightening and straightforward, so omitted.

Now for the definition of K_1 - this simpler than K_0 and we do not have to assume that A is unital. We let $M_\infty(A)^\sim$ be the involutive normed algebra got by adjoining an identity element to $M_\infty(A)$. U(A) denotes the group of unitaries of $M_\infty(A)^\sim$ and U_1 the connected component of 1 in U. This is a normal subgroup of U and we let $K_1(A)$ denote the quotient group U/U_1.

One should think of $K_0(A)$ as an "index" group - many Fredholm-type indices have their values in some $K_0(A)$. Specifically one should think of [e] as the "dimension" of the projection e. In some ways K-theory is like a generalized Fredholm index theory.

Here are a few random examples of K-groups:

1. $K_0(C) = Z$, $K_1(C) = 0$.

2. $K_0(B(H)) = 0$, $K_1(B(H)) = 0$.

3. If O_n is the C*-subalgebra of B(H) generated by n $(n > 1)$ isometries $S_1, \ldots, S_n$ such that $S_1S_1^* + \ldots + S_nS_n^* = 1$ then O_n is simple (i.e. it has no proper closed two-sided ideals) and $K_0(O_n) = Z/(n-1)$. Thus the K-groups can have torsion.

Before listing the basic properties of K-theory we look at some of the applications of the theory.

An *AF-algebra* is a C*-algebra having an increasing sequence of finite-dimensional C*-subalgebras $A_1 \subset A_2 \subset \ldots$ such that the union $U(A_n : n = 1,2, \ldots)$ is dense in A. Some examples are c_0, K(H), and the CAR-algebra so important to mathematical physicists. This class of algebras is diverse and extensive - for example there are uncountably many non-isomorphic simple AF-algebras. The AF-algebras exhibit typical C*-algebra behaviour and are highly non-trivial in general - they are usually not even type I.

If A is an AF-algebra and I is a closed two-sided ideal in A then I and A/I are AF-algebras. It is natural to enquire if the converse is true, i.e. if A is a C*-algebra and I a closed two-sided ideal in A such that both I and A/I are AF-algebras is A an AF-algebra? L. Brown answered this affirmatively in one of the first applications of K-theory. The essential idea was to show that one can lift projections from the quotient algebra A/I to A, i.e. to show that every projection in A/I is the image of a projection in A under the quotient map π from A to A/I. The proof used the 6-term exact sequence of K-theory (this sequence will be exhibited below).

Although not originally conceived in K-theoretic terms it was soon realized that the classification of AF-algebras due to O. Bratteli and G. Elliott (1978) involved K_0 ($K_1(A) = 0$ for any AF-algebra A). For simplicity we'll restrict ourselves to unital AF-algebras. One can define a translation-invariant partial ordering on $K_0(A)$ (for A any unital AF-algebra) by defining the positive cone $K_0(A)^+$ to be the set of all $[e]$ for e a projection in $M_\infty(A)$. Thus $K_0(A)$ becomes a partially ordered group. Now the C*-algebras A and B are said to be *stably isomorphic* if the C*-tensor product of K(H) and A is *-isomorphic to the C*-tensor product of K(H) and B - loosely speaking this means that A and B have the same representation theory. Stably isomorphic C*-algebras have the same K-groups. One of the elegant results of Bratteli and Elliott is that unital AF-algebras A, B whose K_0-groups are isomorphic as partially ordered groups, are stably isomorphic. Moreover if there exists a partially odered group isomorphism ϕ from $K_0(A)$ to $K_0(B)$ such that $\phi[1_A] = [1_B]$ then A and B are actually *-isomorphic!

One can use $K_0(A)$ to investigate the structure of the AF-algebra A. For example, there is a bijective correspondence

between the lattice of closed two-sided ideals of A and the lattice of "ideals" (certain subgroups) of $K_0(A)$. Also, one can give an abstract characterization of the partially ordered groups that can appear as $K_0(A)$ for some AF-algebra A (Effros-Handelmann-Shen 1980). These groups are called *dimension groups*. Let us mention in passing an application of this to Quantum Mechanics: one can use this theory to construct C*-dynamical systems with a given set of temperatures for KMS states. However one of the most striking applications of the theory was its use in solving a long-standing open problem posed by I. Kaplansky in 1958, namely is there a simple C*-algebra, other than C, with no non-scalar projections? B. Blackadar (1980) constructed a certain dimension group having an unusual automorphism property, and this was reflected in an unusual automorphism property of the corresponding AF-algebra. He then used this AF-algebra to construct a simple C*-algebra with no non-trivial projections. Nevertheless this still left open a conjecture of R. Kadison that a certain C*-algebra $C^*_{red}(F_2)$ (a stock counterexample in many situations) had no non-trivial projections. This question was resolved affirmatively by M. Pimsner and D. Voiculescu using K-theory. A Connes gave an alternative proof again using K-theory, but also using his "non-commutative Differential Geometry" (another new exciting area in C*-algebra theory). By the way K-theory has been successfully applied in classical Differential Geometry, to the Novikov Conjecture on the homotopy invariance of higher signatures.

Now it is time to list some of the basic properties of K-theory:

1. If $\alpha : A \to B$ is a *-homomorphism of C*-algebras then there are corresponding group homomorphisms $K_j(\alpha) : K_j(A) \to K_j(B)$ for $j = 0,1$. This defines a pair of covariant functors from the category of all C*-algebras to the category of all abelian groups.

2. (Continuity) If A is an inductive limit in the category of C*-algebras, $A = \lim_\lambda A_\lambda$ say, then $K_j(A) = \lim_\lambda K_j(A_\lambda)$, $j=0,1$.

3. (Homotopy Invariance) If ϕ_t $(0 \leq t \leq 1)$ is a continuous path of *-homomorphisms from the C*-algebra A to the C*-algebra B (in the topology of pointwise convergence) then $K_j(\phi_0) = K_j(\phi_1)$ $j = 0,1$.

4. Stably isomorphic C*-algebras have isomorphic K-groups.

5. (Bott Periodicity) Let B be the C*-tensor product of the C*-algebra A and $C_0(R)$. Then $K_1(A) = K_0(B)$ and $K_0(A) = K_1(B)$.

6. (Periodic Exact Sequence) If I is a closed two-sided ideal in the C*-algebra A then there is an exact sequence

$$K_0(I) \to K_0(A) \to K_0(A/I) \to K_1(I) \to K_1(A) \to K_1(A/I) \to K_0(I)$$

The boundary map δ_1 from $K_1(A/I)$ to $K_0(I)$ can be thought of as a sort of generalized Fredholm index.

5 and 6 are deep and powerful theorems.

A FEW CONCLUDING REMARKS

The theory of extensions and K-theory have been synthesized into a new theory, KK-theory. Readable accounts of K-theory and extensions are to be found in [1], [2] and [3]. Extensive bibliographies are to be found in [1] and [3].

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DIFFERENTIAL EQUATIONS

NIHE DUBLIN

A meeting on *Differential Equations* will take place between 27th - 29th May at the National Institute for Higher Education, Dublin. There will be special sessions on nonlinear equations and on asymptotics for linear equations. Invited speakers include K. Brown (Heriot-Watt), M.S.P. Eastham (King's College, London), H. Ockendon (Oxford), J.R. Ockendon (Oxford) and R.B. Paris (CEA - Euratom). Presentations on any aspect of differential equations (pure or applied) are welcome. Further information can be obtained from Prof. A. Wood, School of Mathematical Sciences, National Institute for Higher Education, Dublin, Dublin 9, Republic of Ireland.

BANACH SPACE ULTRAPRODUCTS

E. Coleman

INTRODUCTION

This note presents a useful tool in Banach space theory: *ultraproducts of Banach spaces*. These provide a uniform method for manufacturing locally similar Banach spaces. In this way they relate local (finite dimensional) and global (infinite dimensional) structure.

Prerequisites are in Section 1; Section 2 contains definitions. Section 3 sketches some typical applications in the local theory of Banach spaces. The conclusion mentions other areas in which ultraproducts are profitably employed. Results for which no reference is given can be found in [4] and [1] which include bibliographies.

1. FILTERS AND ULTRAFILTERS

To set up and handle ultraproducts of Banach spaces effectively, one requires some basic facts about filters and ultrafilters on sets.

Let I be a non-empty set and $\mathcal{P}(I)$ be the power set of I. A *filter* on I is a subset F of $\mathcal{P}(I)$ such that:

F1 $\emptyset \notin F$.

F2 $A \in F$, $B \in F$ imply $A \cap B \in F$.

F3 $A \in F$, $A \subset B \subset I$ imply $B \in F$.

An *ultrafilter* on I is a maximal (proper) filter on I. Equivalently, U is an ultrafilter on I iff (1) U is a filter on I and (2) for all $X \in \mathcal{P}(I)$, $X \in F$ iff $I-X \notin F$.

A trivial application of Zorn's lemma shows that every filter on I can be extended to an ultrafilter on I.

The following topological property of ultrafilters forms the basis of the definition of the Banach space ultraproduct.

THEOREM 1.1. *Let K be a compact Hausdorff topological space; let I be a non-empty set and U be an ultrafilter on I. Then, for each family* $(x_i)_{i \in I}$ *in K, there exists a unique point* $x \in K$ *such that, for every neighbourhood V of x,*

$$\{i \in I : x_i \in V\} \in U.$$

The point x is called the *limit of* $(x_i)_{i \in I}$ *with respect to* U, and is denoted by $\lim_U x_i$.

2. ULTRAPRODUCTS OF BANACH SPACES

Let $\{(E_i, \|\ \|) : i \in I\}$ be a family of Banach spaces over $\mathbb{C}$ (or $\mathbb{R}$) indexed by the set I. U is an ultrafilter on I.

Define Π_0 and N_U as follows:

$$\Pi_0 := \{(x_i)_{i \in I} : x_i \in E_i, \sup_{i \in I} \|x_i\| < \infty\}$$

$$N_U := \{(x_i)_{i \in I} : (x_i)_{i \in I} \in \Pi_0, \lim_U \|x_i\| = 0\}.$$

Note that, for $(x_i)_{i \in I} \in \Pi_0$, $\lim_U \|x_i\|$ exists and is unique by theorem 1.1.

Let $\|\ \|$ be the supremum norm on Π_0:

$$\|(x_i)_{i \in I}\| := \sup_{i \in I} \|x_i\|.$$

Then $\ell_\infty((E_i)_{i \in I})$ is the Banach space $(\Pi_0, \|\ \|)$ over $\mathbb{C}$. It is easy to check that N_U is a closed subspace of $\ell_\infty((E_i)_{i \in I})$.

The *ultraproduct* of the family $((E_i, \|\ \|) : i \in I)$ modulo U is the quotient space $\ell_\infty((E_i)_{i \in I})/N_U$ with the canonical quotient norm, and is denoted $(E_i)_U$ or $\prod_{i \in I} E_i/U$. $(E_i)_U$ is called a *Banach space ultraproduct*; in the case where $E_i = E$ for all $i \in I$ $(E)_U$ is also written E^I/U and is termed the *Banach space ultrapower* of E modulo U.

It is convenient and customary to denote elements of $(E_i)_U$ by $(x_i)_U$ so that

$$(x_i)_U := (x_i)_{i \in I} + N_U.$$

Notice that the quotient norm on $(E_i)_U$ is given by the equation:

$$\|(x_i)_U\| := \inf_{n \in N_U} \|(x_i)_{i \in I} + n\| = \lim_U \|x_i\|.$$

For each ultrapower E^I/U of E there is a canonical isometric embedding $\bar{i}$ of E into E^I/U:

$$\bar{i}(x) := (x_i)_U \quad \text{where } x_i = x \text{ for all } i \in I$$

$$\|\bar{i}(x)\| = \lim_U \|x_i\| = \|x\|.$$

If E is finite dimensional, then E and E^I/U are isometrically isomorphic. The closed balls of E are compact so that for every bounded family $(x_i)_{i \in I}$ in E the limit $\lim_U x_i$ exists in E (by theorem 1.1) and $\|\lim_U x_i\| = \lim_U \|x_i\|$ so that the map $(x_i)_{i \in I} \to \lim_U x_i$ is a linear surjection with kernel N_U, hence induces an isometric isomorphism of E^I/U and E.

The following proposition introduces the theme of the structure-preserving properties of ultraproducts.

PROPOSITION 2.1. *The following classes of Banach spaces are closed under ultraproducts:*

(i) Banach algebras;

(ii) C^* algebras;

(iii) $C(K)$-spaces;

(iv) L^p spaces.

The class of JB^* triple systems is closed under ultrapowers.

PROOF. To prove (i) and (ii) define the natural multiplication and involution on $(E_i)_U$:

$$(x_i)_U \cdot (y_i)_U := (x_i y_i)_U \qquad (x_i)_{U*} := (x_i^*)_U.$$

For (iii) note that $C(K)$-spaces are C^*-algebras and hence ultraproducts of $C(K)$-spaces are $C(K)$-spaces by the Gel'fand-Naimark theorem; (iv) requires the representation theorem for L^p spaces.

Finally, if $(E, \|\ \|, \phi)$ is a JB^* triple system (cf. [2]), then there exists $M > 0$ such that for all $x,y,z \in E$

$$\|\phi(x,y)(z)\| \le M\|x\|\ \|y\|\ \|z\| \qquad (**)$$

so that $(\ell_\infty((E)_I), \|\ \|, \phi)$ is a JB^* triple system with

$$\phi((x_i)_{i \in I}, (y_i)_{i \in I}) := (\phi(x_i, y_i))_{i \in I}.$$

N_U is a J^* ideal in $\ell_\infty((E)_I)$ by (**) and hence $(E^I/U, \|\ \|, \phi)$ is a JB^* triple system.

3. ULTRAPOWER PRINCIPLES AND SUPER-PROPERTIES

One of the successful typical applications of Banach space ultraproducts is in the local theory of Banach spaces, i.e. the study of the finite dimensional structure of Banach spaces and its relation to global structure. In particular, *finite representability* - the most important concept of the local theory - has a simple powerful ultrapower characterisation.

Let E and F be Banach spaces. F is *finitely representable in* E iff

for all $\varepsilon > 0$, for every finite dimensional subspace M of F, there exists a finite dimensional subspace N of E with dim N = dim M, and an isomorphism ϕ from M onto N such that

$$(1-\varepsilon)||x|| \le ||\phi(x)|| \le (1+\varepsilon)||x|| \quad \text{for all } x \in M.$$

The isomorphism ϕ is termed a $(1+\varepsilon)$ isomorphism. For orientation here are two results.

PROPOSITION 3.1.

(i) *Every Banach space is finitely representable in itself.*

(ii) *Finite representability is transitive.*

(iii) *Every Banach space is finitely representative in* ℓ_∞, *in* c_o, *and in the separable reflexive Banach space* $(\prod_{n \in N} \ell_n^\infty)_p$, *the* ℓ_p*-sum of the family* $\{\ell_n^\infty : n \in N\}$ *where* ℓ_n^∞ *is* C^n *with supremum norm* $(1 < p < \infty)$.

The easy proof is omitted. Incomparably deeper is:

THEOREM 3.2 (Dvoretzky). ℓ_2 *is finitely representable in every infinite dimensional Banach space.*

The advertised characterisation of finite representability is as follows:

THEOREM 3.3. F *is finitely representable in* E *iff there exists an ultrafilter* U *on a set* I *such that* F *is isometric to a subspace of* E^I/U.

PROOF. In the format of an expository note there is space just to isolate one characteristic feature of the proof of 3.3 which occurs in the choice of the index set I and the construction of the ultrafilter U on I.

Let I be the set of all pairs (M,ε) where M is a finite dimensional subspace of F and $\varepsilon > 0$. Partially order I by $<$: $(M_1,\varepsilon_1) < (M_2,\varepsilon_2)$ iff $M_1 \subset M_2$ and $\varepsilon_1 \ge \varepsilon_2$. Associate a filter A with $<$ on I:

$I_0 \in A$ iff $I_0 \subset I$ and there exists $(M_0,\varepsilon_0) \in I$ with

$$I = \{(M,\varepsilon) \in I : (M_0,\varepsilon_0) < (M,\varepsilon)\}.$$

Extend A to an ultrafilter U on I.

Since F is finitely representable in E, for each $i = (M_i,\varepsilon_i) \in I$, there exists a $(1+\varepsilon_i)$ isomorphism ϕ_i from M_i onto $N_i \subset E$:

$$(1-\varepsilon_i)||x|| \le ||\phi_i(x)|| \le (1+\varepsilon_i)||x|| \quad \text{for all } x \in M_i.$$

Define a mapping $J : F \to E^I/U$

$$Jx := (x_i)_U, \quad x_i = \begin{cases} \phi_i(x) & \text{if } x \in M_i, \\ 0 & \text{otherwise} \end{cases}$$

J is the required linear isometry.

Note in particular that E^I/U is finitely representable in E for any ultrafilter U on a non-empty set I.

3.2 and 3.3 imply that the modulus of convexity of a uniformly convex infinite dimensional Banach space is dominated by the modulus of convexity of ℓ_2.

Ultrapower techniques allow one to deduce information on the global structure of E from its local structure. The reformulation of local principles results in corresponding ultrapower principles. One of the best examples of this process is:

THEOREM 3.4 (Ultrapower Principle of local Reflexivity) *Let* E *be a Banach space. There exist an ultrafilter* U *on a set* I *and a mapping* J *from* E** *into* E^I/U *such that*

(i) J *is an isometric embedding of* E^{**} *into* E^I/U.

(ii) $J|_E$ *is the canonical embedding* $\bar{i}$ *of* E *into* E^I/U.

(iii) $J(E^{**})$ *is a norm-1 complemented subspace of* E^I/U, *i.e. there is a projection* P *of norm 1 onto* $J(E^{**})$.

PROOF. The Ultrapower Principle of Local Reflexivity is derived from the Principle of Local Reflexivity: (**) for all finite dimensional subspaces $M \subset E^{**}$, $N \subset E^*$ and $\varepsilon > 0$, there is a $(1+\varepsilon)$ isomorphism ϕ from M into E such that

(1) $\phi|_{M \cap E} = Id|_{M \cap E}$.

(2) $\langle f, \phi(x)\rangle = \langle x, f\rangle$ for all $x \in M$, $f \in N$.

Now proceed as in 3.3, taking I to be the set of all triples (M,N,ε) partially ordered with an ultrafilter U on I. Use (**) to define a mapping $J : E \to E^I/U$

$$Ju := (x_i)_U, \quad x_i = \begin{cases} \phi_i(x) & \text{if } x \in M_i, \\ 0 & \text{otherwise.} \end{cases}$$

Parts (i) and (ii) follow.

To complete the proof, define a mapping $Q : E^I/U \to E^{**}$ $Q((x_i)_U) := \lim_U x_i$.
Note that by 1.1 and the weak * compactness of the closed balls of E^{**} the limit is well-defined (identifying E with its canonical image in E^{**}). Set $P := J \circ Q$ to obtain the required projection of norm 1 onto $J(E^{**})$.

One corollary of 3.4 is this: if B is a class of Banach spaces which is closed under ultraproducts and contractive projections (P is a contractive projection iff $P^2 = P$ and $||P|| \leq 1$) then B is closed under formation of biduals, i.e.

$$E \in B \text{ implies } E^{**} \in B.$$

The class of JB* triple systems is closed under contractive

projections, so from 2.1 one deduces the following recent theorem of S. Dineen [2]:

THEOREM 3.5. *Let* $(E, || \ ||, \phi)$ *be a JB* triple system. Then the bidual* $(E^{**}, || \ ||, \phi)$ *is a JB* triple system.*

Intuitively, a *local property* of Banach spaces is a property P such that if E has P, then every Banach space locally similar to E also has P. Super-properties are the mathematically precise explication of this intuition. Let P be any property of Banach spaces. E has the property *super*-P iff every Banach space finitely representable in E has the property P. P is called a *super-property* iff whenever E has P then E has super-P.

Examples of super-properties include: uniform convexity, super-reflexivity, the properties "E is finitely representable in G" and "G is not finitely representable in " for arbitrary fixed Banach space G.

The super-properties of infinite dimensional Banach spaces can be ordered in a hierarchy: there is a weakest (trivial) super-property W, the first (non-trivial) super-property C, and the strongest super-property H. Their definitions run:

H(E) : E is a Hilbert space.

C(E) : c_0 is not finitely representable in E.

W(E) : E is infinite dimensional.

3.1 and 3.2 show that the following implications hold: $H(E) \Rightarrow Q(E) \Rightarrow C(E) \Rightarrow W(E)$ where Q is any super-property. 3.2 implies too that W is equivalent to the super-property D;

D(E) : ℓ_2 is finitely representable in E.

There are many characterisations of C. Here is a recent one deriving from results in [3]. Let BD be the property:

BD(E) : every bounded domain in the complex Banach space E is biholomorphically equivalent to a finite product of irreducible complex Banach manifolds.

Then C is equivalent to super-BD.

Immediate consequences of the hierarchy of super-properties are:

(1) Hilbert spaces possess every super-property;

(2) if ℓ_2 fails to have a given super-property Q, then no infinite dimensional Banach space has Q;

(3) if E is infinite dimensional with even one (non-trivial) super-property, then c_0 is not finitely representable in E.

4. CONCLUSION

The Banach space ultraproduct was developed initially in an interaction of functional analysis and mathematical logic. Thus it is not surprising to find Banach space analogues of theorems of first-order model theory: downward Loewenheim-Skolem theorem, Keisler-Shelah theorem ([8], [1]). A simple corollary of these results is a version of the Banach-Mazur theorem:

<u>COROLLARY 4.1</u>. *Assuming the continuum hypothesis, there exists a Banach space of density character $\aleph_1$ which contains (isometrically isomorphic copies of) every Banach space of density character at most $\aleph_1$. In fact, there is an ultrapower of c_0 satisfying 4.1.*

Recent applications of ultraproduct techniques to non-linear classification problems can be found in [7].

Finally, the material of Section 2 can be generalized to define ultrapowers of locally convex spaces [5], [6].

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ON THE EXISTENCE OF MAXIMAL LOWER BOUNDS

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A *lower semilattice* is a partially-ordered set in which each two elements possess a maximum lower bound or infimum, and a routine induction argument shows that in such a system every finite set also possesses an infimum. Many partially-ordered sets, of course, fail to behave so nicely. To mention one classic example, we can impose a natural partial order on the four dimensional space-time continuum of special relativity by saying that one 'event' (x,y,z,t) precedes another (x',y',z',t') whenever light from the first could reach the 'place' of the second at or before the 'time' of the second, thus:

$$(x,y,z,t) \leq (x',y',z',t') \iff$$

$$\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \leq c(t'-t)$$

where the positive constant c represents the speed of light. It can be shown that in this structure the set of lower bounds of two events (their "common history", so to speak) never possesses a maximum element, except in the trivial case where one of the events precedes the other. There is, however, in this example and in many others, an abundance of *maximal* lower bounds: the common history of two events is 'inductive' in the sense described below. This note arises from an investigation of maximal lower bounds for two or more elements; in particular it concerns the failure of the analogue of the result referred to in the first sentence above: inductiveness of the set of lower bounds for each *two* elements does not imply the same property for *three*.

Following Birkhoff [2] let us call a non-null subset A of a partially-ordered set $(E,\leq)$ *inductive* when to each point x of A there corresponds a maximal point m of A satisfying

$x \leq m$. To avoid possible confusion we should point out immediately that this meaning of the term differs from that of Bourbaki ([3], page 154), who applies it to subsets A of a partially-ordered set such that every chain in A has an upper bound in A. Of course, Zorn's lemma readily shows such a set to be inductive in the Birkhoff sense. To disprove the converse, take E as the set of points (x,y) in the coordinate plane satisfying both $-1 < x-y < 1$ and $x+y \leq 1$, with the coordinatewise partial order described by

$$(x,y) \leq (x',y') \text{ if and only if } x \leq x' \text{ and } y \leq y'.$$

Then E (as a subset of itself) is "Birkhoff-inductive" since each of its members (x,y) lies under the maximal member $((1+x-y)/2, (1-x+y)/2)$, but is not "Bourbaki-inductive" since the chain $\{(x,y) \in E : x = 0\}$ has no upper bound in E.

Now consider the following condition $\mu_L(\alpha)$, applicable to $(E,\leq)$, α denoting a cardinal number:

> for each non-null subset B of E having cardinality at most α, the set L(B) of all its common lower bounds is non-null and inductive. .. $\mu_L(\alpha)$

We term $(E,\leq)$ a $\mu_L(\alpha)$*-system* if it satisfies this condition. If α and β are two cardinal numbers satisfying $\alpha < \beta$ then it is immediate that $\mu_L(\beta)$ implies $\mu_L(\alpha)$; we here exhibit an example to show that in general the converse implication is never valid: that is, that the conditions $\mu_L(\alpha)$ are all logically distinct.

<u>PROPOSITION 1</u>. *Let $\alpha \geq 2$ be a given cardinal number. There is a partially-ordered set satisfying condition $\mu_L(\alpha')$ for every cardinal number α' less than α, but not satisfying condition $\mu_L(\alpha)$.*

PROOF. Denote the set of positive integers by N. Take an index set Λ of cardinality α, a set D of cardinality $\alpha . \aleph_0$ comprising the distinct elements d_n^a ($a \in \Lambda$, $n \in N$), a set X of cardinality α comprising the distinct elements x^a ($a \in \Lambda$), and a set C of cardinality $\aleph_0$ comprising the distinct elements c_n ($n \in N$), where D, X and C are pairwise-disjoint. On the set $E = C \cup D \cup X$ we define a partial order by specifying, as follows, the strict lower bounds of each of three typical elements c_n, d_n^a and x^a:

(i) c_n › each of $c_1, c_2, \ldots, c_{n-1}$, while c_1 is minimum in E;

(ii) d_n^a › each of $c_1, c_2, \ldots, c_n$;

(iii) x^a › d_n^e for all n and for all $e \in \Lambda \setminus \{a\}$ and also c_n for all n.

It may be helpful to refer to Fig. 1, which is a diagrammatic representation of this construct in the case where $\alpha = 3$. Note in particular that

$$\text{if } z \in C \cup D \text{ then } L(z) \text{ is finite.} \qquad (*)$$

Now Let α' be a cardinal less than α, and B a non-null subset of E whose cardinality β is at most α'; three cases may arise:

(I) $B \subseteq X$: then by (*), L(B) is finite, and therefore inductive.

(II) $B \subseteq X$ and $\beta = 1$, that is, $B = \{x^a\}$ for some a: then $L(B) = L(x^a)$ is trivially inductive.

(III) $B \subseteq X$ and $2 \leq \beta \leq \alpha' < \alpha$: then $L(B) \subseteq C \cup D$ and there is $e \in \Lambda$ such that $x^e \in X \setminus B$. Hence $d_n^e \in L(B)$ for all $n \in N$; so while each member of $D \cap L(B)$ is maximal in L(B), each c_n in L(B) lies under the maximal member d_n^e of L(B), and L(B) is again inductive.

The condition $\mu_L(\alpha')$ is therefore satisfied. On the other hand, $\mu_L(\alpha)$ is not: for X is a subset of E having cardinality

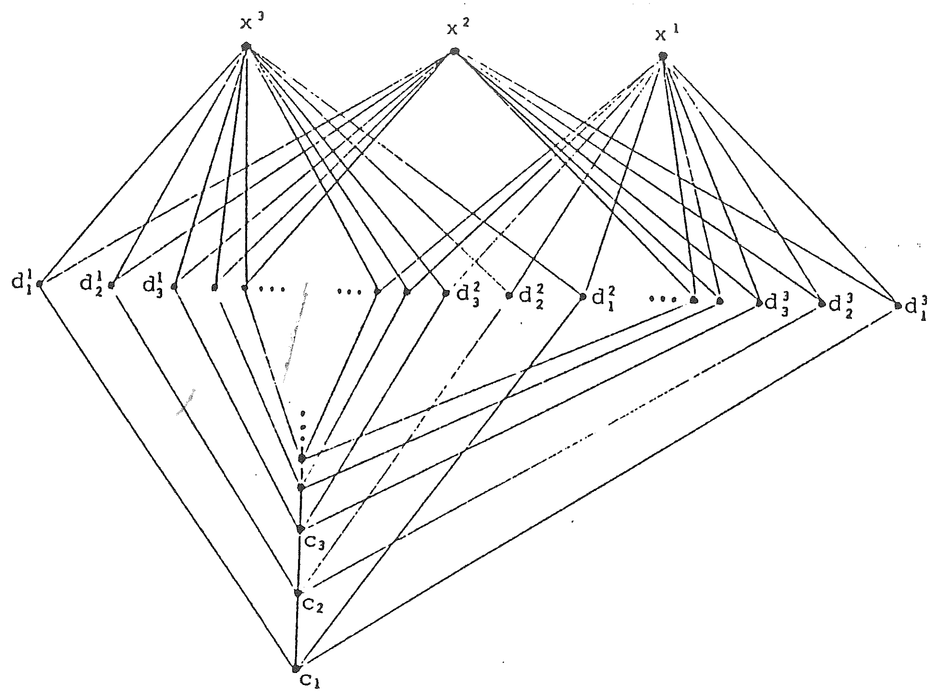

FIGURE 1

α, and $L(X) = C$ has no maximal element.

We can, however, obtain further positive connections between the conditions $\mu_L(\alpha)$ either by imposing some additional condition on the partial ordering, or by insisting that it be compatible with a suitable topology. Note the following definitions: $(E,\leq)$ is *down-directed* if each two of its elements have at least one common lower bound, a subset of $(E,\leq)$ is called *diverse* if no two elements of it are commensurable, a *decreasing* subset D of E is one for which $x \leq y$ and $y \in D$ together imply $x \in D$, and a partially-ordered topological space $(E,\leq,\tau)$ is termed T_1-*ordered* [4] if, for each of its elements x, both L(x) and the set M(x) of all the upper bounds of x are closed sets.

LEMMA. *If $(E,\leq)$ has no infinite diverse subsets, then its inductive decreasing subsets are precisely the finite unions of sets of the form $L(x)$.*

PROOF. Clearly such a set is inductive decreasing. For the rest, it suffices to note that if D is an inductive decreasing subset, then $D = \bigcup(L(m) : m \in M)$ where M is the set of maximal points of D; and that M, being diverse, is here finite.

PROPOSITION 2. *A $\mu_L(2)$-system without infinite diverse subsets is a $\mu_L(n)$-system for every positive integer n.*

PROOF. For each pair of points x,y in the $\mu_L(2)$-system $(E,\leq)$ denote by $\eta(x,y)$ the set of maximal lower bounds of x and y, so that

$$L(x,y) = \bigcup(L(m) : m \in \eta(x,y)).$$

Observe that for any z in E we have

$$L(x,y,z) = \bigcup(L(n) : n \in \eta(m,z),\ m \in \eta(x,y));$$

and if each of the (diverse) sets $\eta(m,z)$, $\eta(x,y)$ is finite, this (by the lemma) is inductive and $(E,\leq)$ is a $\mu_L(3)$-system. The obvious induction extends this argument to establish the proposition.

PROPOSITION 3. *Let $(E,\leq,\tau)$ be down-directed, compact and T_1-ordered; then it is a $\mu_L(\alpha)$-system for every cardinal $\alpha \geq 1$.*

PROOF. Let B be a non-null subset of E. The family $\{L(b) : b \in B\}$ of closed subsets of compact E has the finite intersection property, and therefore

$$\emptyset = \bigcap(L(b) : b \in B) = L(B)$$

(compare [6], Theorem 1). Let z be any point of $L(B)$, and C a chain in $M(z) \cap L(B)$; again, the family $\{M(c) \cap L(B) : c \in C\}$ of closed subsets of (closed, and therefore compact) $M(z) \cap L(B)$

has the finite intersection property, whence C has an upper bound in $M(z) \cap L(B)$; an application of Zorn's lemma now shows that $M(z) \cap L(B)$ has a maximal point, which is then maximal in $L(B)$ and lies over z: thus $L(B)$ is inductive.

REMARKS

Bearing in mind the power and the widespread use of maximality arguments in many areas of mathematics, there are surprisingly few references in the literature to the ideas here presented. The only major investigation seems to be that of Benado (see, e.g. [1]) who explored in detail what we have here termed $\mu_L(2)$-systems (without the assumption of down-directedness) but not $\mu_L(3)$ or beyond. The present writer's involvement is due to an attempt to generalize the idea of a *topological semilattice* - by which is meant a semilattice equipped with a topology such that the map taking each pair of elements to their infimum is continuous. If in an arbitrary down-directed partially-ordered topological space one considers continuity of the map taking each n-tuple of points to the set of *all* their common lower bounds, having first made a sensible choice of topology for the ranges of these maps, one gets a hierarchy of conditions (for varying n) each of which specializes to "continuity of the infimum" in the semilattice case. It transpires (see [5]) that the conditions $\mu_L(n)$ are convenient for obtaining satisfactory product theorems concerning such bound-continuity conditions. The exploration of these conditions is still incomplete: for example, no full understanding of when a sub-(order/topological)-system inherits them has been obtained, and $\mu_L(n)$-systems may well have a role to play in this matter also.

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THE USE OF BERNSTEIN POLYNOMIALS IN CAD/CAM .. BEZIER CURVES

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INTRODUCTION

In this article we present an application of the Bernstein approximation theorem to CAD/CAM (Computer Aided Design/Computer Aided Manufacture) and to Computer Graphics. In these fields we need to be able to model complex shapes. Indeed, the method was originally proposed by P. Bezier and was used to model surfaces in automobile design for the French firm Regie Renault (cf. [1]). Many other techniques are available for modelling curves and surfaces such as B-splines, cubic splines, standard polynomial interpolation, parabolic blending etc. A useful book on these topics is [2] (with code in BASIC). We include Bezier curves here for two main reasons: first, their practical value, and, second, their roots in classical approximation theory.

MATHEMATICAL BACKGROUND

We start with the Weierstrass approximation theorem, which states roughly that one can approximate a function by a polynomial. Note that the theorem is non-constructive in the sense that neither the statement of the theorem nor its proof allows us to construct the polynomial. The result goes as follows.

Assume that $f(t)$ *is a continuous function on the interval* $[0,1]$. *Given* $\varepsilon > 0$, *there exists an integer* $N > 0$ *and a polynomial* $P(t)$ *of the same degree such that*

$$|f(t) - P(t)| < \varepsilon \quad \text{for all} \quad t \in [0,1]. \qquad (1)$$

This result forms the basis of much numerical analysis, e.g. numerical integration and interpolation, finite element analysis etc. We note that the polynomial P which satisfies (1) is not necessarily of interpolating type.

Bernstein (1912) actually constructed a polynomial which satisfied the conditions of the theorem. The result is:

Let f(t) be a continuous function on the interval [0,1]. *Define the* nth *degree polynomial* P(f;t) *by*

$$P(f;t) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j} f(j/n). \qquad (2)$$

Then the polynomials P(f;t) *converge uniformly on* [0,1]; *this means that given* $\varepsilon > 0$, *there exists an integer* N *such that for all* $n \geq N$ *we have*

$$|f(t) - P(f;t)| < \varepsilon \quad \text{for all} \quad t \in [0,1]. \qquad (3)$$

For a proof of this result, see [3].

BEZIER CURVES

In representation (2) we define the so-called *control points*

$$p_j = f(j/n) \quad \text{for} \quad j = 0, \ldots, n, \qquad (4)$$

and the so-called *blending functions*

$$B_{j,n}(t) = \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j}. \qquad (5)$$

In this case we can write the resulting curve as a polynomial of degree n as follows:

$$P(t) = \sum_{j=0}^{n} p_j B_{j,n}(t). \qquad (6)$$

Notice that (6) is a vector equation: let $p_j = (x_j, y_j, z_j)$, $j = 0, \ldots, n$, be the coordinates of the control vertices and suppose that $P(t) = (x(t), y(t), z(t))$. Then from (6) we have

the following:

$$x(t) = \sum_{j=0}^{n} x_j B_{j,n}(t)$$

$$y(t) = \sum_{j=0}^{n} y_j B_{j,n}(t)$$

$$z(t) = \sum_{j=0}^{n} z_j B_{j,n}(t)$$

These last three equations form the basis for any computer implementation of Bezier curves. Input for such a program would be the control points and the number of subdivisions of the interval $0 \leq t \leq 1$. This last parameter will basically determine the number of points on the newly generated Bezier curve. In most cases the code would be written in FORTRAN.

REMARKS

1. We can think of a Bezier curve as being associated with the vertices of a polygon which uniquely define the curve slope. Only the first and last vertices of the polygon actually lie on the curve; however, the other vertices define the derivatives, order and shape of the curve (see Fig. 1).

2. By changing the control vertices we can change the resulting Bezier curve and this property gives a good intuitive feeling for the CAD/CAM designer.

3. The number of polygon vertices fixes the order of the resulting polynomial which defines the curve and furthermore, the Bernstein basis has a global span, i.e. the values of the blending functions given by (5) are nonzero for all parameter values over the entire span of the curve. Thus, changing a control vertex changes the entire curve. This eliminates the possibility of producing local change. These problems can be overcome but one must resort to the so-called B-splines (cf. [4].

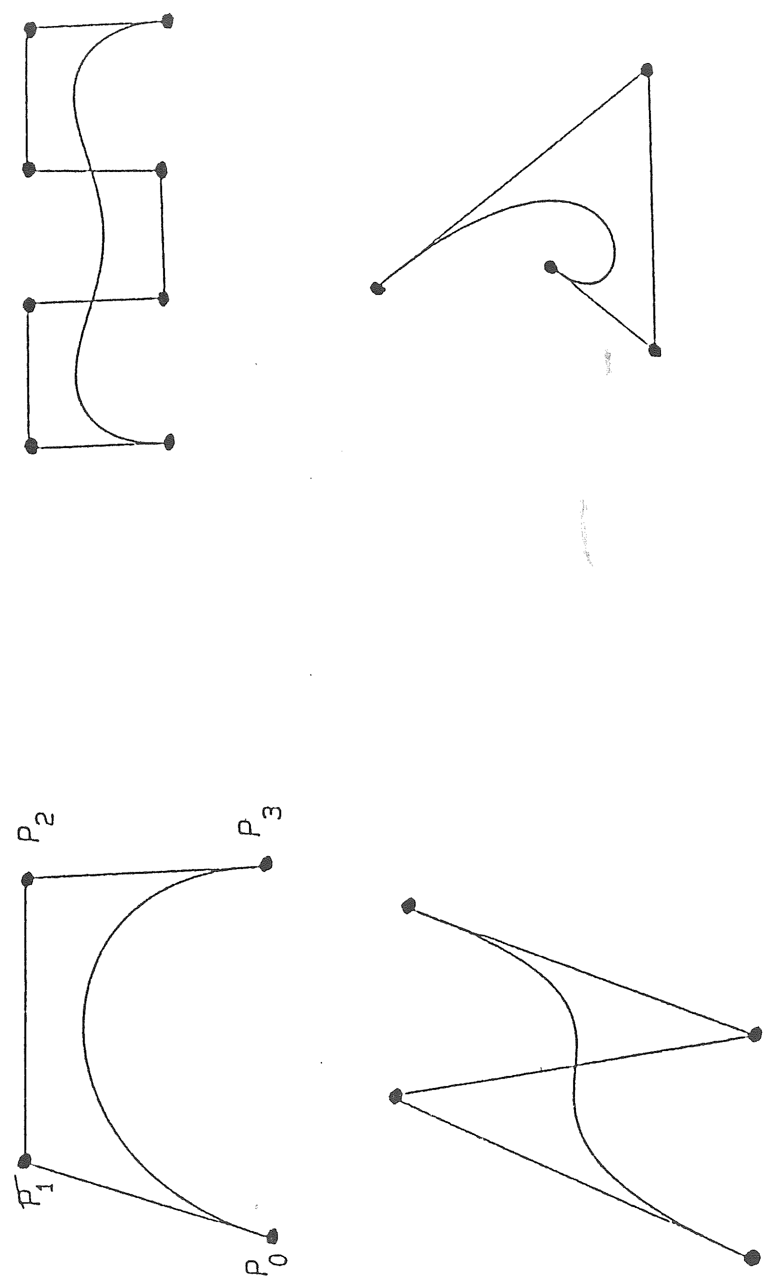

FIGURE 1: Some Bezier Curves and Associated Control Vertex Polygons

BEZIER SURFACES

Equation (6) can be generalized to three-dimensional surfaces by generating the Cartesian product of two Bezier curves. The resulting Bezier surface can be written as (again, a vector equation):

$$P(t,s) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} B_{i,n}(t) B_{j,m}(s) \qquad (8)$$

In this case we have to input (n+1) x (m+1) control points (p_j). For an implementation of these surfaces, see [2], pp. 230-231.

CONCLUSION

We are only able to give a short review of one topic in a fast growing area. Many other techniques exist for approximating curves and surfaces and new methods are being constantly developed. For a good introduction to Computer Graphics, see [4], in particular pp. 309-331.

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A CONNECTED TOPOLOGY WHICH IS NOT LOCALLY CONNECTED

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To demonstrate that a connected topological space may not be locally connected, authors of modern text-books on point-set topology usually employ an example, either of a geometrical nature in the real plane (such as the so-called "topologist's sine curve". and "infinite broom"), or of a number theoretical nature in the integers (such as the "relatively prime integer topology"), or of an analytical nature in the real line (such as the "indiscrete or pointed extensions of the reals" and the "one-point compactification of the rationals"; see [1]). The complete exposition of such an example tends to rely heavily on a knowledge of the various intrinsic properties of the supporting set. For the instructor who may, perhaps, prefer a more abstract and topologically succinct example, an alternative is readily available.

Let X be an infinite set containing distinct points x,y. A topology τ for X may be defined by declaring open, apart from $\emptyset$ and X itself, those subsets G of X for which $y \notin G$ *and* either $x \notin G$ or X-G contains (at most) a finite number of points. Observe that $\tau = (\gamma \cup \varepsilon(x)) \cap \varepsilon(y)$, where γ denotes the well known cofinite topology for X and $\varepsilon(x)$, $\varepsilon(y)$ denote, respectively, the excluded point topologies $\{G \subseteq X : x \notin G\} \dot{\cup} \{X\}$ and $\{G \subseteq X : y \notin G\} \cup \{X\}$ (see [1]). That is, τ is the intersection of a Fort topology $\gamma \cup \varepsilon(x)$ and an excluded point topology $\varepsilon(y)$.

It is immediate that (X,τ) is a connected space (since $\tau \subseteq \varepsilon(y)$ and $(X,\varepsilon(y))$ is obviously a connected space). Let U be any proper τ-open neighbourhood of x. Thus $y \notin U$ and X-U is finite. If $z \in U$, $z \neq x$, then $\{z\}$ and $U-\{z\}$ are each τ-open (since y belongs to neither, $x \notin \{z\}$ and $X-(U-\{z\}) = (X-U) \cup \{z\}$ is finite) so that U is not τ-connected. It follows that (X,τ) is not locally connected (at x).

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NONCOMMUTATIVE ANTICOMMUTATIVE RINGS

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An associative ring R is said to be *anticommutative* if $xy + yx = 0$ for all $x,y \in R$. If R has characteristic 2, then the concepts of commutativity and anticommutativity coincide, but Z_3, with the usual addition and trivial multiplication, shows that an anticommutative ring need not have characteristic 2.

If a ring R satisfies $x^2 = 0$ for all $x \in R$ then clearly R is anticommutative, but not conversely. However, if R is anticommutative it is easy to verify that R satisfies each of the following identities.

(i) $2x^2 = 0$ (ii) $(xy - yx)^2 = 0$ (iii) $x^2y - yx^2 = 0$.

Frequently, when looking at commutativity theorems for rings, one requires counterexamples to show that certain conditions are not sufficient for commutativity. For example, if $(xy)^2 = x^2y^2$ for all $x,y \in R$ and either of the following conditions holds then R is commutative:

(a) R has unity; (b) R has no non-zero nilpotent elements.

To show that some such additional condition is necessary, it is enough to produce a non-commutative ring in which $x^2 = 0$ for all $x \in R$. In this note, for finite rings, we pose the question, *"what is the order of a smallest noncommutative anti-commutative ring?"* and show that the answer is 27. Since this number is odd, we see that it is also the answer to the question, *"what is the order of a smallest noncommutative ring satisfying the identity $x^2 = 0$?"*.

First of all we produce a ring of order 27 with the desired properties. Let $A = (a_{ij})$ be the ring of those 4x4 matrices with entries in the field Z_3, such that $a_{ij} = 0$ if $j \leq i$, $a_{23} = 0$, $a_{24} = a_{13}$ and $a_{34} = -a_{12}$. Then it is easily checked that R is a noncommutative anticommutative ring of order 27. In more abstract terms, R can be expressed as follows: if C_n is the cyclic group of order n and $\oplus$ denotes the direct sum of groups, then $(R,+) \cong C_3 \oplus C_3 \oplus C_3 = \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle$, where $a^2 = b^2 = c^2 = ac = bc = ca = cb = 0$, $ab = -ba = c$ determines the multiplicative operation in R.

We proceed to show that no ring of order less than 27 can be both noncommutative and anticommutative, so let R be a ring with these properties. Since every finite ring is the direct sum of rings of prime-power order and since a direct sum of rings is anticommutative if and only if each of its direct summands is anticommutative, we may confine our attention to rings of prime-power order. If (R,+) is cyclic, then R is commutative - this eliminates rings of prime order and if $|R| = p^2$ for some prime p, we may assume $(R,+) \cong C_p \oplus C_p$. Clearly, we may also eliminate rings of characteristic 2. Thus we need only consider the following values of $|R|$ with corresponding structures for (R,+):

(i) $|R| = 8$, $(R,+) \cong C_2 \oplus C_4$;

(ii) $|R| = 9$, $(R,+) \cong C_3 \oplus C_3$;

(iii) $|R| = 16$, $(R,+) \cong C_2 \oplus C_8$, $C_2 \oplus C_2 \oplus C_4$, $C_4 \oplus C_4$;

(iv) $|R| = 25$, $(R,+) \cong C_5 \oplus C_5$.

We can eliminate 9 and 25 using the following result.

LEMMA. *If p is an odd prime, then $C_p \oplus C_p$ cannot be the additive group of a noncommutative anticommutative ring.*

PROOF. Let R be a counterexample and let $(R,+) = \langle a\rangle \oplus \langle b\rangle$. Since R is anticommutative, $x.x + x.x = 0$ for all $x \in R$, so $x^2 = 0$, since $|R|$ is odd. If $ab = 0$ then $ab + ba = 0 \Rightarrow$

$ba = 0 = ab$, so R is commutative, a contradiction. Suppose that $ab = ra + sb$ where $r,s \in Z_p$. Then $a^2b = ra^2 + sab$, so $sab = 0$, and so $s = 0$. Finally $ab = ra$, so $ab^2 = rab = 0$ which gives $ab = 0$, a contradiction.

Next, we suppose that $(R,+) \cong C_2 \oplus C_4$ or $C_2 \oplus C_8$ and $R = \langle a\rangle \oplus \langle b\rangle$, where b has order 4 or 8. In either case, $2ab = (2a)b = 0$, so $2ab = ab + ba$, and R is commutative. The case $(R,+) \cong C_2 \oplus C_2 \oplus C_4$ is dismissed in a similar manner. We are left with the possibility that $(R,+) \cong C_4 \oplus C_4$. Suppose that $(R,+) = \langle a\rangle \oplus \langle b\rangle$ where $4a = 4b = 0$. Consider first the case where $a^2 = b^2 = 0$. Then we get a contradiction, as in the proof of the lemma. Thus we may assume that one generator (a say) satisfies $a^2 \neq 0$. Since $2a^2 = 0$, $a^2 \in \{2a, 2b, 2a+2b\}$, the set of elements of order 2 in R. Suppose first that $a^2 = 2a$ and let $ab = ra + sb$, where $r,s \in Z_4$. Then $2ab = a^2b = a(ab) = ra^2 + sab = 2ra + sab$. This gives $(sr)a + s(s-2)b = 0$. Hence s is even and if $s \neq 0$, r is even. This implies that ab has order 2 and so R is commutative, a contradiction. Thus $s = 0$ and $ab = ra$, $r = \pm 1$. Then $ab^2 = (ab)b = rab = r^2a = a$, so $2ab^2 = a(2b^2) = 0 = 2a$, a contradiction.

Finally, we may suppose that $a^2 = 2b$, since if $a^2 = 2a+2b$ we may replace b in the basis by a+b. If $ab = ra + sb$, we get $(rs)a + (2r+s^2)b = 0$. Thus s is even and if $s \neq 0$, r is even also, so ab has order 2, a contradiction. Hence $s = 0$, so $2r + s^2 = 0$, r is even, $2ab = 0$ and we are finished.

Let S be the ring of order 32 where $(S,+) = \langle a\rangle \oplus \langle b\rangle$ $C_8 \oplus C_4$, with $a^2 = 4a$, $b^2 = 2b$, $ab = -ba = 2a$. Then S is a noncommutative anticommutative ring of order a power of 2. By our previous analysis, S is a smallest such 2-ring and in addition, S is a smallest such ring of even order. Finally, we observe that S is a smallest ring of the desired type such that (S,+) is a 2-generator group.

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DYNAMICAL LIE ALGEBRAS

Allan Solomon

An algebra is just a vector space with multiplication. A *Lie Algebra* has a peculiar form of multiplication referred to as commutation - concretely realized by matrices x and y

$$[x,y] = xy - yx,$$

together with the axioms following from this. There is no loss of generality in considering all finite dimensional Lie algebras as just algebras of matrices.

There seem to be at least two reasons why Lie algebras play a prominent role in modern theoretical physics:

1. *The Symmetry Aspect*: Theories which undertake to provide a description of space and time start by emphasising the underlying symmetry, before going on to give more detailed discussions of the mechanics. Thus if we say "Empty space looks the same everywhere" this is interpreted as "Translational Symmetry" (and such symmetries do in fact lead to observable conservation laws). The mathematical formulation of this type of symmetry leads to a structure called a *Lie Group*, and Lie algebras are related to Lie groups in much the same way as the additive properties of logarithms relate to the multiplicative properties of numbers.

2. *The Dynamical Aspect*: When we get down to giving a more detailed mechanical description of nature, we must provide dynamical laws, such as Newton's laws for Classical Mechanics. The latter are expressed in differential calculus form, so it is fairly obvious that the techniques of differential calculus will prove useful in a study of classical mechanics. One formulation of the basic laws of Quantum Mechanics, however, utilizes the Lie algebra commutator introduced above - for example the famous Heisenberg relation for position operator q and

momentum operator p (and unit operator I)

$$[q,p] = ihI$$

(where h is Planck's constant over 2π). If we take (q,p,I) as a basis in a (complex) vector space, then the Heisenberg relation (together with $[q,I] = [p,I] = 0$) gives us a Lie algebra structure (the Heisenberg algebra). It is not surprising, therefore, that Lie algebras play a pivotal role in a discussion of quantum dynamics.

It is in their second role, that of providing a dynamical description of quantum theory, that Lie algebras have become recently increasingly fashionable, as *Dynamical Lie Algebras*; this is distinct from their use in the first context, that of Symmetry Lie Algebras. To the pure mathematician, of course, there is no distinction, it is the same Lie algebra structure involved - merely the application which differs. In fact, the same (isomorphic) Lie algebra may be used in two different contexts, in one case as a symmetry algebra and the other as a dynamical algebra.

The simplest example of the preceding discussion is the algebra $so(3)$. This is defined as the set of 3x3 (real) antisymmetric matrices. As a (real) vector space, it is three-dimensional and we can choose a basis (J_1, J_2, J_3), where $[J_1, J_2] = J_3$ and the two similar commutation relations obtained by cyclic permutation also hold. This algebra is well known as the Lie algebra of the group of rotations in three-dimensional space. Rotational symmetry is, of course, an assumed symmetry of the world, and so this exemplifies the symmetry use of Lie algebraic theory. Any rotation matrix R may be written $R = \exp J$, where $J \equiv a_1J_1 + a_2J_2 + a_3J_3$ is an element of $so(3)$; this makes R automatically orthogonal - and J is essentially the logarithm of R.

However, this same algebra also occurs as a dynamical algebra in, for example, the theory of superconductivity. Such quantum systems are specified by writing down a hamiltonian operator H; eigenvalues of H, for example, give the set of energy levels of the system. In the case of a simple model of superconductivity - the so-called BCS model named after the Americans J. Bardeen, L.N. Coopes and J.R. Schrieffer who developed it in 1957 - this hamiltonian may be written as

$$H = a_1\hat{J}_1 + a_2\hat{J}_2 + a_3\hat{J}_3.$$

(Here the real numbers a_1, a_2 and a_3 are related to the kinetic and potential energy of the system.) Now of course the $\hat{J}_i$ are no longer 3x3 matrices; they are operators. But they have precisely the same commutation relations as the J_i above of *so*(3). Thus H may be considered as an element of *so*(3). Note that this algebra is *not* a symmetry algebra of H; that is, H does not commute with the $\hat{J}_i$ (that is, $[H,\hat{J}_i] \neq 0$) and it does not commute with the "rotations" $\hat{R} = \exp \hat{J}$ corresponding to the $\hat{J}_i$ (that is, $\hat{R}H\hat{R}^{-1} \neq H$). However, Lie algebraic techniques can now be utilized to solve the BCS problem. We know that a 3x3 anti-symmetric matrix may be diagonalized by an orthogonal matrix. This purely algebraic result can be extended to any dimension, and indeed to the operator H above. We may thus find explicitly a "rotation" R such that, for example,

$$\hat{R}H\hat{R}^{-1} = a J_3, \; (a^2 = a_1^2 + a_2^2 + a_3^2).$$

Since J_3 may be chosen to be a diagonal operator, the spectrum of $\hat{R}H\hat{R}^{-1}$, and thus that of H, is immediate. This gives the energy levels of the system.

The most obvious use of these dynamical algebras - to obtain the spectrum of a quantum system - gives them the name Spectrum Generating Algebras. (The corresponding Lie groups are often referred to as Dynamical Groups.) Their first use in Elementary Particle Physics dates from the work of Y. Ne'eman and collaborators in the early sixties; subsequently the present writer employed the method in condensed matter physics (superfluidity, 1970) and there has recently been a resurgence in nuclear physics (the Interacting Boson Model) as well as in

straightforward potential theory. Current applications of the technique are to quantum systems exhibiting many coexisting phases simultaneously - such as superconductivity and magnetism.

I have given no references in this short, informal and introductory note; Arno Bohm and Yuval Ne'eman of Austin, Texas are currently editing a review monograph ("Spectrum Generating Algebras and Dynamical Groups", World Scientific, 1987) which will cover the history and applications of the subject.

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MATHEMATICAL EDUCATION

SOME THOUGHTS ON THE ROLE OF MATHEMATICS*

P.F. Hodnett

1. VARIOUS ROLES OF MATHEMATICS

Mathematics appears in a variety of guises as a language, an analytical tool, a vocation.

Mathematics is the language of quantities, size, order, shape. The need to communicate in a quantified manner requires a mastery of the language, i.e. mathematics. Hence the traditional requirement for engineers and physicists to be educated mathematically and with more subjects moving towards quantification, e.g. biology, social science, psychology, economics, management science, etc. there is a growing requirement for a mathematical education for a larger section of the professional community than has been the case in the past.

Mathematics is used as an analytical tool for example by engineers to create a model of a physical system which for instance may be represented by a differential equation. The solution of the equation which yields understanding of the behaviour of the model and hence the physical system requires knowledge of certain mathematical techniques such as the Laplace Transform, Fourier Series, Bessel functions, Legendre functions, etc. As more subjects become quantified different mathematical techniques are becoming important. For example, the techniques of Operations Research are now important in quantifying management science as are the techniques of probability and combinatorics in relation to social science.

* This is a revised version of a talk given at the NCEA Seminar on Science and Computing Education in NCEA designated institutions held in Tralee, RTC, 12 April, 1985.

1.1 MATHEMATICS - A VOCATION

Mathematics is a vocation when it is employed as a body of technique in solving business and industrial problems. Vocational programmes in Applied Mathematics exist at NIHE, Limerick and one in Applied Mathematical Science at NIHE, Dublin. These programmes have vocational objectives. The objective of the programme at Limerick is to produce graduates with developed analytical skills and an ability to model real industrial or business systems as situations suitable for quantitative analysis and optimization. To achieve these aims the course includes basic elements of Business Studies and Engineering Science both to facilitate effective communication with colleagues whose training is in the theory of business or the practice of engineering and to ensure that a realistic model is created of the system under investigation. Expertise in modelling is developed through modules in System Theory, Operations Research and Industrial Engineering. Analytical and computational skills are developed through modules in Mathematics, Statistics and Computer Science. A project in the final year of the course provides the opportunity to integrate the different elements in the programme in modelling and analysing a real industrial or business problem. Of the three graduating classes to date (at Limerick) totalling approximately forty students all received ready employment in a range of positions in industry and business. The type of job and type of employer ranged over management information systems at a multinational company; statistical analysis for a market survey firm; software development for a computer manufacturer; software development for a manufacturer of electrical equipment; financial services in a Semi-State company; accountancy in a chartered accounting firm; production planning in a multinational company; actuarial work with an insurance company; quality control in a manufacturing company; a variety of computer programming and software development positions in the computer industry.

This encouraging employment record is similar to the situation in the US where current demand for applied mathematics graduates is growing at such a rate that starting salaries now rival those for electrical engineers and exceed those for business graduates. The supply of graduates from those universities with established applied mathematics programmes such as Brown University, Rensselaer Polytechnic Institute and New York University does not satisfy the demand and other universities such as Clemson University have revised their mathematics curriculum towards applications of mathematics so as to prepare students for roles in business and industry.

Employers range from companies like AT & T and General Motors with large research and development sections to Standard Oil Co. of California where the Mathematics and Statistics Consulting group cut the cost of testing new locomotive oil additives from $240,000 to $3,000 by showing that one short engine test was statistically equivalent to the results of 40 longer tests previously employed. Other companies who previously did not employ mathematicians now realize according to the Recruiting Officer for the large distribution company Foremost-McKessian Inc. that mathematics is "a universal language for attacking problems". One problem this company faces is to devise the most cost-effective way to adjust distribution networks in response to an increasingly fast-changing market place.

The programme at NIHE, Dublin which has not yet graduated students has a particular emphasis on mathematical modelling. Both the programmes at Limerick and Dublin have vocational goals in contrast to those programmes which are concerned primarily with the study of mathematics.

2. CHANGING EMPHASIS IN MATHEMATICS

In many countries and examples are the UK and the US, the emphasis of the recent past on the study of mathematics as a self-contained subject is receding and the traditional connection between mathematics and its application is being restored. This movement has occurred with the advent of computers since now realistic models of problems can be formulated and analysed with the aid of a computer. Of course the move away from the study of mathematics as an autonomous discipline was also not unconnected to the declining enrollment in such courses since students found the courses unattractive and employers were uninterested in recruiting graduates from the courses. Coincident with this change in content of mathematics programmes there are appearing new technologies requiring different mathematics. For example, the mathematics which is the language of computer science consists of set theory, equivalence relations, ordering, Boolean algebra, logic networks, graph theory, combinatorics. These pure mathematics subjects of the recent past have found new applications. Interestingly much of this material is also relevant to the quantification of behavioural and social science subjects such as sociology, psychology and management science. Developments in electronics relating to digital systems require the language of discrete mathematics rather than the continuous mathematics of electrical and mechanical engineering and hence require difference rather than differential equations and Z-transforms in place of Laplace transforms.

3. SUBJECT QUANTIFICATION

There is a move towards quantification of subjects such as biology, sociology, psychology, psychiatry, management science etc. where the fundamental dynamics of the underlying systems are not yet properly understood. For example in 1986 the 4th Conference on the Mathematical Theory of the Dynamics of Biological Systems took place in the UK. The 26th British Theoretical Mechanics Colloquium at Leeds in March, 1985 heard an invited lecture by J.D. Murray (an applied mathematician) on "A New Approach to the Generation of Biological Pattern and Form". There is now a Centre for Mathematical Biology at Oxford University. J.D. Murray emphasises that for mathematics to contribute to the quantification of other subjects it is necessary for mathematicians to invest substantial time in attempting to understand the subject to which the mathematics is to be applied. Mathematicians must develop an understanding of other scientific subjects in order to interact fruitfully with scientists in quantifying subjects which were hitherto described in qualitative terms. To play a role in the move towards quantification in other subject areas, mathematicians must actively seek new areas of application for mathematics.

It is frequently found that the mathematical models which arise in the new quantification of subjects are models which have previously been studied in relation to other subjects. Also the simplest models often give most insight. For example the equation of simple harmonic motion (describing the motion of a linear spring) is

$$\frac{d^2x}{dt^2} + n^2x = 0 \qquad (1)$$

When equation (1) is altered to

$$\frac{d^2}{dt^2} + n^2\sin x = 0, \qquad (2)$$

it describes the motion of a simple pendulum (or a non-linear spring). Equation (2) can be transformed into

$$\frac{d^2x}{dt^2} + ax + bx^3 = 0, \qquad (3)$$

where a,b are constants. When two terms are added to equation (3) so that it becomes

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + ax + bx^3 = f\cos wt, \qquad (4)$$

where k, f and w are constants, the equation is called Duffing's equation and has been widely studied in relation to the vibrations of mechanical systems. This same equation has recently been used in modelling aspects of the behaviour of the

brain (see [1]). It is, perhaps, not too surprising that Duffing's equation might model diverse phenomena since it is one of the simplest yet realistic models of movement away from an equilibrium state with d^2x/dt^2 representing acceleration, $ax + bx^3$ representing the inherent attractive force (non-linear) of the system towards equilibrium, $k\frac{dx}{dt}$ representing the resistance of the system to movement (i.e. inertia) and $f \cos wt$ representing a periodic external force attempting to create movement.

4. INTEGRATION OF MATHEMATICS WITH APPLICATIONS

In the past mathematics and its applications was integrated in a cohesive whole. In the more recent past this integration was broken with an overemphasis on the language and notational aspects of mathematics with consequent adverse effects of an educational and vocational nature. An example (in this author's view) of such an adverse educational result is the presence of set theory as a notational device in the Leaving Certificate mathematics syllabus without other material in the syllabus to which set theory can be applied. This can lead students to view mathematics only as a language divorced from applications.

In the vocational or training sense the study of mathematics involves concentration on accuracy and emphasis on the fact that there is only one correct solution to a properly posed mathematical model. This approach involves in-depth study and detailed analysis and constrasts sharply (i) with the reality of attempting to create mathematical models where sometimes there is uncertainty about the input data (for example in economic models) and (ii) with the case where a crude model is most appropriate either because an approximately correct answer (e.g. 70% - 80% correct) is sufficient or a rapid answer is required. Students therefore need to experience the necessity of producing crude simple models which yield approximately correct but rapid results. Otherwise, there is a danger that overemphasis on accuracy and 100% correct answers will lead to the production of graduates with a need to analyse every problem in depth before producing any answer.

The relative absence of a large research and development sector in the industry of this country means the absence of what is in other countries a traditional source of employment for mathematically trained graduates. However, the development of the scientific approach to management in this as well as in other countries also requires the analytical abilities which mathematically trained graduates possess. There is and will be an increasing demand from this sector for mathematicians who can model and analyse complicated management and industrial problems.

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COMPUTERS AND THE MATHEMATICS CURRICULUM

Eoghan MacAogain

INTRODUCTION

Information technology will have a radical and pervasive effect on education, affecting both the aims, content and teaching and learning methods of all subjects; it will also affect the organisation of education, enabling its wider dispersal, both in terms of location and age of pupils. Teaching and learning methods, assessment and the curriculum are all bound up together, but here attention will be focused on curricular matters, specifically on the impact of computers on the mathematics curriculum, concentrating on the senior cycle of second-level education. Many of the ideas, however, will have a broader application, and will apply to mathematics education in general.

COMPUTERS AND MATHEMATICS

Firstly, a few comments on this perennial topic for debate. Although there are many connections between computers and mathematics, for example, programming may be regarded as a branch of logic (see e.g. Murphy [9]), present opinion is almost unanimous in regarding the linking of mathematics and computer studies on the curriculum as undesirable. For example: "Their view was unanimous that computer studies should not be regarded as part of mathematics but should ideally exist within a separate department." (Cockroft Report [1], par. 397.) Reasons commonly given are the need for special training in order to teach computer studies, the need to prevent such a subject becoming an elitist, and in particular a sexist one, and the fact that the linking of computer studies with mathematics inhibits its development across subject boundaries.

In Ireland, computer studies is an optional section in the mathematics syllabus in the senior cycle; as a result, its hours either come out of the mathematics allocation, or are tacked on to the end of the normal timetable. Of those teachers involved with computing in schools, mathematics teachers form the largest group. The situation is similar in Britain. So we see that *de facto* there is a strong connection between computer studies and mathematics in the education system. That this is bad for computer studies is generally recognised. However, I believe that it is equally bad for mathematics, because the time and resources of both teachers and pupils are diverted from mathematics. The situation should improve at second level with the establishment of computer studies as a subject in its own right, as seems likely.

CALCULATORS

It is instructive to consider briefly the impact which calculators have had on pupils' learning of mathematics. The commonest objection to the introduction of calculators into the mathematics classroom was that their use would impair pupils' abilities to carry out pencil and paper calculations. In fact the opposite has proved to be the case. In a meta-analysis of 79 research reports in the USA, Hembree and Dessart [3] report: "At all grades but Grade 4, a use of calculators in concert with traditional mathematics instruction apparently improves the average student's basic skills with paper and pencil, both in working exercises and in problem solving." They also report that the use of calculators improves a student's attitude towards, and self-concept in, mathematics. The use of calculators has been allowed in the Leaving Certificate since 1986.

COMPUTER EDUCATION IN IRISH SECOND-LEVEL SCHOOLS

The situation as it evolved up to 1983 was described in a previous article by Moynihan [8]. Some developments since then will be briefly described. Since September 1980 computer studies has been an optional module in the mathematics syllabus in the senior cycle, requiring 35 hours of study, but not forming part of the Leaving Certificate. In September 1984 an optional computer studies module was introduced into the junior cycle. This programme is independent of mathematics, requires about 70 hours of study and is not examinable. A syllabus committee for this module was set up, and it completed its work in May 1985. Both syllabi are available in the Department of Education's Rules and Programme for Secondary Schools 1986/87. Since August 1986 the Curriculum and Examinations Board has taken over responsibility for syllabus committees. At the time of writing, the Board of Studies for Science, Technology and Mathematics to the Curriculum and Examinations Board is still preparing its report. It seems likely that it will recommend a computer studies module in the junior cycle, independent of mathematics, and the establishment of a full senior cycle subject called either computer studies, or, more generally, information technology. Some pilot projects are being supported by the Department of Education: topics include courseware development and control applications. The spread of information technology across subject boundaries has been slow. A national policy on information technology has often been called for, but is slow in coming. In particular, there seems to be no commitment to adequate pre-service and in-service teacher training. However several third-level institutions, including Trinity College, the Regional Technical College Waterford, and Thomond College/NIHE Limerick provide postgraduate courses in computer education for teachers. On the hardware side, the official commitment to Apple has been maintained, although Commodores, BBCs and Amstrads are also popular; however, the days of the 8-bit machine must be numbered. Languages being used include BASIC, COMAL and LOGO. An interesting feature of the junior cycle syllabus is the recommendation of PROLOG as a suitable descriptive language.

GENERAL TRENDS IN MATHEMATICS EDUCATION

The effects of the widespread availability of microcomputers on the mathematics curriculum are difficult to predict, but the following trends can be discerned:

(a) the increased use of numerical methods; where these are already in use, their earlier introduction, either before, or at the same time as, analytical methods.

(b) the increased use of graphical methods, including dynamic graphics.

(c) the use of symbolic manipulation systems (for example, to carry out algebraic manipulation, or differentiation). These systems are not new, but have not up until recently been available to teachers.

(d) an increased emphasis on algorithmic thinking, linked to programming, and in particular, on the development, rather than just the use, of algorithms.

(e) the displacement of low-level skills, such as solving a simple equation, by higher-level skills, such as interpreting and applying the solution.

(f) the possibility of giving a dynamic, as well as a static view of some mathematical topics.

(g) the use of more realistic numbers in applications of mathematics: this should lead to a reduction in the level of abstractions.

(h) the formal teaching of estimation skills: mental arithmetic, approximation and general 'number sense'.

(i) an increased emphasis on problem-solving methods such as guess and check or successive approximation.

(j) the possibility of more heuristic learning: the learning of mathematics through personal discovery can be facilitated through the use of 'tool-kits' or 'microworlds'.

These challenge the linear model of learning.

(k) an increased use of mathematical modelling and simulation.

SOME TOPICS AT SENIOR-CYCLE, SECOND-LEVEL

In a previous article, Seda [11] has discussed some topics which might form a basis for the discrete component in a better balanced curriculum for mathematics students. Many of these topics are also suitable for senior cycle pupils: some candidates for inclusion would be graph theory, mathematical logic, difference equations and the study of algorithms. For a description of an algorithm option in the A-level examination, see Kowszun [6].

In analysis, we often have a choice between continuous and numerical methods: the basic concepts of analysis, e.g. rate of change and area, can be implemented in either discrete or continuous ways: difference and sum, or derivative and integral. The discrete concepts are simpler, but their implementation generally involves much calculation. Calculus is limited in the range of functions it can deal with. Nevertheless, it is a powerful and beautiful theory. Continuous analysis is necessary for the theoretical justification of, and as a guide to the numerical methods (see Winkelmann [13] for a discussion of this topic). Numerical techniques may be introduced either before, or at the same time as, analytical ones. Symbolic differentiation and integration programmes will become available to teachers and pupils: less time will need to be spent on techniques or tricks-of-the-trade, leaving more time available for developing theoretical insight. Graphical techniques may be used to facilitate understanding of the concepts involved in analysis. Much useful graphics software already exists.

Symbolic manipulation systems (e.g. muMATH) will have an impact on the teaching of algebra similar to that of calculators on the teaching of arithmetic. Algebraic manipulations may now be carried out by machine. The implications of this are that less importance will attach to the ability to manipulate algebraic expressions, and more to the ability to formulate them. This will hopefully lead to a better understanding of their meaning. Facilities available in such systems generally include manipulation of polynomials, symbolic differentiation and integration, arbitrary precision arithmetic and simplification of algebraic expressions. Graphics will be available in similar programmes in the near future. General data types are increasingly important: besides numbers, other data types, such as strings, matrices, sets and Boolean variables, should be encountered. Spreadsheet programmes can facilitate the introduction of matrix algebra. The commercial programmes are quite useful, although a spreadsheet specially designed for mathematics would be an improvement. Incidentally, these programmes have many other uses in mathematics (see Hsiao [5] for some examples). Programmes are being developed which help the user to explore algebra: for example, Algebraland, being developed by the Xerox Palo Alto Research Center, keeps a record of the algebraic transformations applied during the solution of an equation, thus facilitating an examination of the solution path. The case has been made by several authors that the teaching of programming facilitates the understanding of many mathematical concepts, and in particular algebraic concepts (see e.g. Hart [2]). The concept of variable appears very early in programming (in most languages). Simple programming can give pupils concrete examples of the use of variables. Of course, a variable in programming is not exactly the same as a variable in mathematics: programming gives a more dynamic, changing view. Programming may also be used to introduce the ideas of operation on a variable, and function. Inverse and composite functions are also met in easily understandable ways; e.g. in LOGO, LT is the inverse of RT. The concept of function may also be compared and contrasted with that of procedure. Procedures commonly take an input, do something with it, and produce an output. More generally, the writing of programs involves the use of formal symbols, and of a particular syntax, and this in itself has analogies with algebra.

The availability of graphics makes certain areas of geometry, for example transformation geometry and three-dimensional geometry, much more easily accessible to pupils. The writing of programmes to produce certain graphic effects also provides a strong motivation for learning geometry. 'Tool-kits' are being produced which enable the user to explore some aspect of mathematics. The most commonly used one at the moment is Papert's Turtle Geometry incorporated in the language LOGO. Another example is the Geometric Supposer (see Schwartz [10]). This enables the user to experiment with geometric constructions, and hopefully to make and test hypotheses. Much work has been done on the impact of LOGO on mathematics education: for a discussion of this, see Hoyles and Noss [4]. Some effects on geometry are: a pupil's experience of angle may be enlarged; LOGO provides a procedural description of curves rather than a static one, opening the question of which curves should be emphasised (see Laski [7]); curves may be defined as limits, or using recursion: either of these techniques is commonly used when defining a circle in LOGO. Both the display devices and the programming languages used have their own intrinsic geometries; for example, BASIC generally uses absolute 2D cartesian co-ordinates, while LOGO uses relative 2D polar co-ordinates (see Oldknow's essay in the Ware conference report [12] for a discussion of this point).

CONCLUSION

There are several main effects of the increased availability of computers on the mathematics curriculum: firstly, there will be pressure for new topics to be introduced into the curriculum; secondly, existing topics can be taught and learnt in new ways: this can change the sequencing of topics, and, more fundamentally, can challenge the concept of a curriculum topic: the curriculum could follow the trend at primary level and become more pupil-centred; thirdly, some pressure is being put on the teaching of mathematics due to the present lack of adequate teacher training in computer studies.

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CURRICULUM DEVELOPMENT IN SECONDARY SCHOOL MATHEMATICS, WITH SPECIAL REFERENCE TO GEOMETRY

P.D. Barry

1. There is a substantial amount of published material on curriculum development in secondary school mathematics, particularly on the "New Mathematics" epoch since about 1950. On the latter, there is a progression from the objectives and syllabi of the pioneers, through the projects and project-evaluations, on to the text-books, to articles reviewing progress or non-progress, and to books. I am certainly no specialist in this field, and the aim of this article is to provide references to what I have encountered (without any expectation of completeness), to review briefly aspects of the international scene, and hopefully foster a consideration of the Irish experience in the light of this.

There is a very informative and readable book, Howson, Keitel and Kilpatrick [19], and surveys in the UNESCO publication *New trends in mathematics teaching*·Vol. III (1972) [42]. These will be my main references, but they and the others listed here contain a host of others. Other references specific to mathematical education are Cooper [8], Howard, Farmer and Blackman [18], and Servais and Vargo [37]. The UNESCO:IBE publication *Curriculum innovation at the second level of education* [11] provides a more general background. There are also chapters on mathematics in general books, e.g. in Tanner [39].

As a background, it would perhaps be well to mention briefly the variety internationally of the modes of control and innovation in education; [19] deals with this and in particular gives a reference (p. 58) to a grouping of European countries having similar administration organisations for education and consequently similar approaches to curriculum development (a first group being characterised as having little decentralisation but

some central government movement towards it, a second group as having little decentralisation but some grassroots movement towards it, and a third as having considerable decentralisation but some myth-making about local autonomy). There is also a reference (p. 78) to a distinction between profuse and confined systems rather than between centralised and decentralised ones (a profuse system containing a variety of development and dissemination agencies and a confined system containing a limited number).

2. To come now to the objectives of the pioneers of the 'new' mathematics, we refer to the College Entrance Examination Board: Commission on Mathematics: *Program for College Preparatory Mathematics* (New York, 1959) [6], the OEEC (Paris) publications *New Thinking in School Mathematics* (1961) [30] and *Synopses for Modern Secondary School Mathematics* (1961) [31], the introductions to Dieudonné [10] and Choquet [5], and others.

A common aim was to bring school mathematics more closely into line with university courses, by introducing topics that have emerged over the last century or so as being of basic importance, e.g. sets, functions, equivalence and order relations, the laws of algebra, vectors. There was a desire to have clear concepts and proper (instead of pseudo) reasoning. The existing material would have to be pruned to make room for the new, and the treatment of it should be efficient and informed with the new spirit. Progress was seen to lie not only in having new syllabi but also in adopting new pedagogical approaches to the presentation of material.

That much was largely common ground and has been carried into effect in the reforms in many countries. For comment on curriculum development strategies, projects, pedagogical approaches, and evaluations of these, we refer to [19].

There was a general expectation among the pioneers that pupils following the new courses would give an improved performance all-round, on the retained material of the old syllabus as well as on the new. For an analysis of the outcome of these expectations we refer to [19, Ch. 7] and to Pidgeon [34, Ch. 7]. From the pressure due to so much new material, and perhaps also because it was felt that the clarity of new concepts and the greater power of the new approaches would suffice, many of the new courses had much less time for and emphasis on practice at acquiring skills at manipulation, solving of problems, and application to other fields. To quote Dieudonné [10, p. 12]:

> "I have swept away all traditional considerations and allowed myself to be guided uniquely by my knowledge of what immediately *follows* a secondary education, namely, the first-year courses in universities (or in the polytechnics)."

For resistance to this trend we refer to Ahlfors *et al* [1] and Nevanlinna [29]. Some references have self-explanatory titles, e.g. Kline [21] *Why Johnny can't add: the failure of the New Math*, Thom [40] *'Modern' mathematics: an educational and philosophical error*, and Vogeli [44] *The rise and fall of the 'New Math'*. There has been continuing controversy over the decline in skills, and lack of application.

3. Having dealt with what was largely common ground, allowing for differences in emphasis and detail, we now turn to an area of great divergence, to wit geometry. Chapter 3 in the UNESCO survey [42] starts with the following:

> "The content for geometric study at the secondary school level has been one of the most controversial issues debated by mathematicians and educators for more than fifty years. Many conferences on this subject have led to two distinguishable positions: one, to preserve a large section of Euclidean synthetic axiomatic geometry; the other, to make a completely new approach to the study of space."

A clear focus to modern controversy on this can be given by quoting Choquet [5, p. 13]:

> "From the mathematician's point of view, the most elegant, mature and incisive method of defining a plane (or space) is as a two (or three)-dimensional vector space over R having an inner product, i.e. a symmetric bilinear from u.v such that u.u > 0 for all non-zero u."

and p. 14:

> ":we have a 'royal' road based on the concepts of 'vector space and inner product.'"

The UNESCO survey [42] went on to detail different basic positions on geometry which we re-summarise as follows.

4(i) The first broad approach we mention is the least integrated. It organises local areas of school mathematics, allows several approaches to a topic, and for pedagogical reasons avoids (resists, in fact) placing these in a globally organised or axiomatised framework.

Examples of this are found in England, with an emphasis on transformation geometry [see, e.g. 27, 36], and in the Netherlands with a course using axial symmetries as a major building block [13].

(ii) The second broad approach we mention has as focus a vector space with inner product, but is divided into three streams.

(a) A first stream envisages an initial geometrical familiarisation stage, with an informal treatment of vectors, plane transformations and geometrical figures.

There is then produced a synthetic axiomatic system which aims at a vector space with inner product as an ultimate goal. Examples of this are due to Papy [33] and Servais in Belgium. Choquet [5] and Queysanne, Revuz etc. [35] in France. The difference between the French and Belgians in this is that the French axiomatisations assume a knowledge of the real number system whereas the Belgian approach integrates a build-up of the real numbers with the geometry.

(b) A second stream also envisages an initial informal geometrical familiarisation stage. Axioms are then given for a vector space with inner product, and the geometry is extracted from this. Thus this type of course starts with a vector space. Examples are due to Dieudonné [10], and [26] from the Strasbourg area of France.

(c) A third stream is based on a familiarisation with the concept of vector space without any motivation from or reference to geometry, e.g. from groups, rings, integral domains and fields of numbers. Then from axioms for a vector space with inner product, the geometry is built up. For advocacy of this approach see Glaymann [16].

This approach (ii) is the most integrated of the three approaches, and makes the most extensive demand for the inclusion of abstract algebra. It involves an initial substantial stage of affine geometry, in which explicitly or implicitly there is distance along each line in a plane but the units of distance on the various lines are not co-ordinated so as to produce distance on the plane. Then at an appropriate stage the geometry is specialised to Euclidean geometry.

(iii) The third broad approach is intermediate between the other two in point of integration. The geometry is Euclidean from the start, synthetic and within a global framework that is or can be axiomatised. This broad approach offers the greatest continuity with the past. One treatment is based on congruence

more or less in the style of Euclid, as completed by Hilbert; examples of this occur in West Germany and the USA. There is also a combination of these two in SMSG axioms [22 or 41] in the USA, and there are axiomatisations based on distance, e.g. [23] and [15].

Within (ii) and (iii) there is also a division between courses which contain an axiomatic organisation from the start of secondary school (age 12), and those which proceed in two phases, an initial organisation of the experience and spatial intuition of pupils with local deductions (age 12-15) followed by a related global axiomatic organisation.

5. The specific geometrical aims of advocates of approach 4(ii) are perhaps most completely expressed by Dieudonné, although it can be seen that the other courses implement what is being articulated by him. Arguments based on congruence and similarity of triangles are to be omitted, and objects such as triangles and parallelograms are to be referred to as little as possible; instead, arguments based on linear algebra are to be used, and an emphasis placed on abstract concepts such as a geometric transformation regarded as a single entity. The trigonometry is to be of rotations rather than angles and we are to avoid [10, p. 11]:

> "those unbelievable complications and fallacies surrounding such a straightforward concept as that of 'angle' when regarded from the traditional point of view."

and further [10, p. 16],

> "As for the so-called 'measurement' of angles, it deservedly wallows in the general confusion which reigns in this sphere."

Approach 4(i) concerns itself with pedagogy as much as syllabus content, and stresses that pupils should be helped to discover mathematical facts and development for themselves, and not have the facts dictated to them. We refer to Freudenthal [14, p. 426].

Those who continue to support approach 4(iii) do so on the basis that to subjugate geometry to linear algebra leads to an impoverishment of geometry. They value the visual as a helpful rewarding method of reasoning, they are reluctant for pedagogic reasons to impose extra unnecessary layers of abstraction on the young, and they value how mathematics can arise naturally in the small in geometry, growing from simple to more complex situations, in contrast with having to deal from the start with a large, abstract, complex system. They query whether 4(ii) is in fact a 'royal road' to geometry, as for example the difficult topic of 'angle' is submerged in the topic of rotations. On this side and ranged mainly against approach 4(ii) we can refer to Nevanlinna [29], Thom [40], and two speakers Osserman [32] and Grunbaum [17] at the Fourth International Congress on Mathematical Education at Berkeley in 1980. To quote the latter briefly:

> "Disparaging the importance of the visual, instinctive - even tactile - aspects of geometry and urging their replacement by tool-oriented techniques certainly will not make the future role of geometry any easier. Such an attitude is inherently as absurd as the promotion of soundless music, or verbal rendering of paintings."

6. Bell [3] gives an idea of the amount of retraining of teachers necessary for a type 4(ii) approach; he says that in France teachers attended in-service training for one afternoon a week for a whole year.

7. What has been described applies to secondary schools. There is another stand which perhaps should be mentioned, although it appertains mainly to primary schools. The educational psychologist Jean Piaget has conducted a major series of experiments on how children learn mathematics and in particular what they are or are not able to assimilate at a given age. This has profound implications for the content and sequence of mathematical topics in primary school. An introduction to Piaget's work is given in Copeland [9]. There is also reference to Piaget's work in [18].

As recounted in Copeland [9, p.7]:

> "The Bourbaki group of mathematicians attempted to isolate the fundamental structures of all mathematics. They established three mother structures: an algebraic structure (the prototype of which is the notion of a group), a structure of ordering, and a topological structure. These were later modified to include the notion of categories.'
>
> At a meeting of mathematicians and psychologists in Paris, Piaget and Dieudonné on listening to each other found that there was a direct link between these basic mathematical structures and Piaget's three structures of children's operational thinking."

So if you encounter, on p. 3 of [19] a calm mention of the possibility of introducing category theory at primary school, this is the likely source.

8. Thus internationally there is great diversity in the treatment of geometry, with continuing controversy. In France and Belgium vector spaces predominate; in the Netherlands and England there is localisation, with an emphasis on transformation; congruence is a continuing component in West Germany; for the USSR we refer to [24]. The USA, the original home of the 'new' mathematics, is a country of great diversity. For one sombre

analysis consider Allen [2] in 1984:

> "Now, our mathematics programs, for all except the very best students, present algebra without structure, geometry without proof, and, worst of all, instruction that, some believe, has 'no established or widely accepted set of goals.'"

A look at the special issue on geometry of the *Mathematics Teacher* [25] in September 1985 shows no evidence of any geometrical approach other than one based on congruence.

Servais and Varga [37] gave syllabi for eight countries. Cooper [8] refers in great detail to Great Britain.

9. Turning now to the Irish scene, I have not encountered much published material. Perhaps it would be well to refer to a general work, Mulcahy [28], and to the fact that the Irish Association for Curriculum Development has published a bi-annual journal *Compass* [7] since 1972.

In the present syllabus for the Intermediate Certificate, which has been current for a generation, we can see the introduction of the mathematical topics which we referred to in Section 2 as being common ground internationally. I do not wish to publish at this stage a detailed analysis of its geometrical content, which to put it mildly is inadequate. Briefly it has emphasised transformations, but the appearance of a vector-space focus as in 4(ii) is misleading, as it is only a veneer; basically the course never departed from proofs by congruence, although it has trophies from other courses, such as equipollence from Papy and angle-measure from Birkhoff.

In the new syllabus, announced on 25 September 1986 and to be first examined in 1990, a traditional treatment of geometry, based on congruence, is being reverted to.

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Forms are available from the Treasurer.

"PARAMETER ESTIMATION FOR STOCHASTIC PROCESSES"

By *Yu A. Kutoyants*

Published by *Heldermann Verlag*, Berlin, 1984. viii + 206 pp.
DM 56. ISBN 3-88538-206-7.

This book is a translation by B.L.S. Prakasa Rao of a completely revised and extended version of a former book of Kutoyants, published in Russian. It has 200 pages divided into five chapters.

Chapter One consists of a quick review of the theory of parameter estimation when the data consist of independent observations. It briefly describes maximum likelihood estimation, Bayesian estimation, the Cramer-Rao lower bound on the mean squared error of an estimator and the less widely known theorem of Hajek establishing the asymptotic minimax lower bound for the risk of an estimator. The treatment here is discursive and clearly assumes a good degree of prior knowledge on the part of the reader.

Chapters Two through Four deal with three different situations where dependent observations arise and where the observations themselves consist of the values of a random function observed at all time points in an interval $[0,T]$. Chapter Two considers observations of the form

$$X(t) = S(\theta,t) + n(t), \quad 0 \leq t \leq T$$

where for each θ, $S(\theta,.)$ is a known function of t, $(n(t) : 0 \leq t \leq T)$ is a gaussian process with mean zero and known covariance function and θ is the parameter to be estimated. Chapter Three considers observations on a diffusion process

$$dX(t) = S(\theta;t,X)dt + \sigma dW(t)$$

where for each θ, $S(\theta;.,.)$ is a known function, σ is a diffusion coefficient, $W(.)$ is the Wiener process and again θ is the parameter to be estimated. The differentials above are defined with respect to the Ito integral. Chapter Four considers observations on a Poisson process $(X(t) : 0 \leq t \leq T)$ with intensity function $S(\theta;.)$ where, for each θ, $S(\theta;.)$ is a known function of t and it is required to estimate θ. The three chapters are similarly organised beginning with the appropriate generalisations of the Cramer-Rao lower bound and Hajek's theorem and continuing on to investigate the consistency, efficiency and asymptotic mean squared error of the maximum likelihood and Bayesian estimates of θ. Given certain smoothness conditions on the function S it is shown that both the maximum likelihood estimate and the Bayes estimate are asymptotically efficient. However in the absence of these smoothness conditions the maximum likelihood estimate and the Bayes estimate have different limiting properties and the Bayes estimate is the only asymptotically efficient estimate, This makes for an interesting divergence from the asymptotic equivalence of maximum likelihood and Bayes estimates monotonously encountered when dealing with independent and identically distributed observations. Examples are given of the applications of these results to situations that arise in signal processing and communications theory.

Chapter Five gathers together several general theorems on the properties of the likelihood ratio-theorems that have been used earlier in the proofs of Chapters 2-4. This is done since many of the proofs are identical or analogous for each of the observation types considered.

I am unable to judge the usefulness of this book from the point of view of an expert in this area. However, for someone familiar with estimation based on independent observations, it offers a clear insight into the difficulties involved in extending some of the results to the case of dependent observations. Perhaps unavoidably, the notation is complex, the proofs are difficult and in many cases draw on obscure (to me at any rate) results in functional analysis. Bedside reading it is NOT!

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"PARTIAL DIFFERENTIAL EQUATIONS FOR SCIENTISTS AND ENGINEERS"

By *G. Stephenson*

Published by the *Longman Group Ltd*, 1984. x + 161 pp.
ISBN 0-582-44696-1.

Dr Stephenson, who is the author of several textbooks on mathematics, has written a compact and eminently readable account of partial differential equations at an elementary level. The book itself is intended for scientists and engineers, and the inclusion of the last name in the title is evidence of the increasing sophistication of the mathematical techniques required of the engineer nowadays.

By normal standards Dr Stephenson's book is small, yet he has succeeded in including a large amount of useful information within its 160 pages. He concentrates mainly upon the so-called equations of mathematical physics, e.g. Laplace's equation, the wave equation, the heat conduction equation and Schrodinger's equation. Occasionally functions involving more than two independent variables are considered. In discussing boundary value problems the author inserts a short section on ill-posed problems.

The book commences with a classification of the second order partial differential equations. Then orthonormal functions are introduced, with a brief but welcome reference to completeness, after which the author moves on to the question of the separation of variables, which could probably be described as the heart of the book. Numerous well-chosen examples of this useful technique are included, while at the same time the reader is introduced to the notion of discrete eigenvalues and eigenfunctions. Sturm-Liouville theory is mentioned, admittedly only briefly, but some substantial results are obtained. Solving Laplace's equation in three dimensions in non-rectangular coordinates gives rise naturally to the so-called special functions, but, as the author is careful to state in his preface, little space is devoted to them and any of their properties which are used are stated without proof. There is also a short but useful chapter on continuous eigenvalues and Fourier integrals.

In a book of this size a chapter on the Laplace transform might be regarded as a luxury, but the author's decision to include one is undoubtedly justified. There are many excellent examples of the application on the Laplace transform, and indeed of the Fourier transform, to boundary value problems involving partial differential equations. There is also an introduction to the Green's function, the Heaviside step function and the delta function.

It is hardly possible nowadays to write a textbook on partial differential equations without some reference to numerical methods and the tremendous impetus given to these methods by the recent explosion in computer technology. One can only agree wholeheartedly with the author's own cogent observation that results which are spurious may be accepted as correct just because they come out of a computer.. Perhaps from a desire to combat this dangerous tendency, the author devotes his last chapter to a brief exposition, mostly by way of examples, of the finite difference and the finite element methods. The Rayleigh-Ritz idea, which relies heavily upon the calculus of variations, is used to illustrate the finite element approach.

Every chapter is followed by a set of problems and answers to all the problems are provided. The book is remarkably devoid of misprints, in fact the only nontrivial one I encountered occurs in the second of Problems 5, page 77, which deals with the generating function for $J_n(x)$. I feel that the author has succeeded admirably in his intention of producing an elementary text which is accessible to any undergraduate student with fairly basic mathematical education and I should have no hesitation in using this book should the occasion arise.

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"POLYHEDRAL COMBINATORICS AND THE ACYCLIC SUBDIGRAPH PROBLEM"

By *M. Junger* (Research and Exposition in Mathematics 7)

Published by *Heldermann Verlag*, Berlin, 1985. x + 128 pp.
DM 36. ISBN 3-88538-207-5.

Combinatorial optimisation [1,2] is the branch of mathematics which tackles such problems as the travelling salesman problem, shortest path problems, matching problems and network flow problems. More specifically, if S is a non-empty finite set and f is a real-valued function on the subsets of S then combinatorial optimisation refers to the problem of maximising f on a given collection of subsets. Since S is finite the most obvious way of solving such a problem is to list all the values of f in question and to pick the largest one. This method is too naive to be of much practical use. Instead, such problems are solved by developing algorithms for finding the required solution. Combinatorial optimisation is a child of the computer age. Not only are computers used to find the solutions, but a number of the problems in the field have arisen

in research in computer design and the theory of computation. There are many "real-world" problems which can be solved by applying the techniques of combinatorial optimisation. (With regard to applications of optimisation techniques in the real world it is a salutory exercise to read the case studies in [3], e.g. "the celebrated brand X washing machine shipping catastrophe", which show how careful one must be in making decisions based on a mathematical model.)

The monograph under review discusses the following combinatorial optimisation problem: given a directed graph D with an integer "weight" on each arc, determine an acyclic subdigraph of maximum weight. An equivalent version of this *Acyclic Subdigraph Problem* (ASP) is the *Triangulation Problem*: find a simultaneous permutation of the rows and columns of a non-negative square matrix such that the sum of the entries above the diagonal of the permuted matrix is maximum. The Triangulation Problem has an application in economics.

Let the digraph D have n arcs. Each subset B of the arcs has a 0-1 n-dimensional incidence vector x_B associated with it. The acyclic subdigraph polytope $P_{AC}(D)$ is the convex hull of all x_B, where B runs over all acyclic arc sets in D. The ASP may then be formulated as the integer programming problem: maximise $c^t x$ subject to $x \in P_{AC}(D)$, given the non-negative vector $c \in Z^n$. The ASP is an example of an NP-hard problem (see [4]) and the idea of associating a polytope with the feasible set of such a problem and then of applying linear programming techniques, has become popular in recent times. It turns out to be crucial to determine the facets ((n-1)-dimensional faces) of the polytope, and the central achievement of this monograph is the determination of several classes of facets of $P_{AC}(D)$. The author expresses the confident hope that "the algorithmic exploitation of our results will in fact lead to the effective solution of large instances of real-world problems which can be formulated as an Acyclic Subdigraph Problem".

The book is well organized and well written and, as well as dealing with the ASP, it gives an excellent survey of polyhedral combinatorics, although the reader may wish to fill in the background by consulting some of the references below. The theory of the book, due to Grötschel, Jünger and Reinelt, was awarded the IBM Computer Application Prize for 1984.

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BOOKS RECEIVED

"MATHEMATICAL FORMULAE" (Fourth Edition)

By *S. Barnett and T.M. Cronin*

Published by *Longman*, Essex, 1986. 77 pp. Stg £3.95.
ISBN 0-582-44758-5

A reference work providing a compact collection of mathematical formulae designed specifically for engineering and science students at university or college. For this fourth edition the authors have added new sections covering such topics as z-transforms, orthogonal polynomials and Walsh functions; other additions include further properties of matrices and a useful list of symbols and notation. The tables of logarithms have been replaced by frequently used statistical tables.

"ON THE EXISTENCE OF NATURAL NON-TOPOLOGICAL, FUZZY TOPOLOGICAL SPACES"

By *R. Lowen*

Published by *Heldermann Verlag*, Berlin, 1985. xvi + 183 pp.
DM 34.00 ISBN 3-88538-211-3

This monograph presents a unified study of several important examples of natural fuzzy topological spaces; the space of probability measures on a separable metrizable topological space, the space of Radon probability measures on a linearly ordered topological space, and the hyperspace of uppersemicontinuous fuzzy sets on a uniform space.

It is shown how these spaces can be canonically equipped with non-topological fuzzy topologies, and in each case the richness of information contained in these fuzzy structures when compared to classical structures is demonstrated.

A first chapter is provided which contains sufficient concepts from abstract fuzzy topology to make the book self-contained.

"ABELIAN VARIETIES" (Second Edition)

By *D. Mumford*

Published by *Oxford University Press*, India, 1986. vii + 279 pp. Stg £7.50. ISBN 0-19-560528-4.

This book is a systematic account of the basic results about abelian varieties. It includes an exposition, on the one hand, of the analytic methods and results applicable when the ground field k is the complex field C and, on the other hand, of the scheme-theoretic methods and results used to deal with inseparable isogenies when the ground field k has characteristic p.

The revised second edition contains, in addition, appendices on "The Theorem of Tate" and the "Mordell-Weil Theorem".

PROBLEM PAGE

First item this time is one of those intriguing problems which is often solved more easily by non-mathematicians! I heard it from Richard Bumby who traces it back to John Conway.

1. Find the next entry in the following sequence:

$$1,\ 11,\ 21,\ 1211,\ 111221,\ 312211,\ \ldots\ .$$

Here is another problem with a simple solution which is not so simple to discover.

2. Find an infinite family of pairs of distinct integers m,n such that:

m,n have the same prime factors, and

m-1, n-1 have the same prime factors.

Now for the solutions to some earlier problems, from March 1986.

1. How long is the recurring block of digits in $(0.\dot{0}0\dot{1})^2$?

I first heard this problem from David Fowler, who uses it as an example to show that simple arithmetic can be surprisingly tricky.

Many people's first guess at the answer is 6 digits or 9 digits, but in fact the recurring block has 2997 digits! To be precise:

$$(0.\dot{0}0\dot{1})^2 = 0.\dot{0}00001002\ldots99699799\dot{9}.$$

In case you think that there is a misprint here, the string 998 is indeed absent.

Once the pattern in this recurring block has been noticed it is not hard to show that

$$(0.\underbrace{\dot{0}0 \ldots 0\dot{1}}_{n})^2 = \frac{1}{(10^n-1)^2}$$

has a recurring block of $n(10^n-1)$ digits. To verify that the decimal expansion has the form

$$(0.\underbrace{\dot{0}0 \ldots 0\dot{1}}_{n})^2 = 0.\underbrace{\dot{0}0 \ldots 0}_{n}\underbrace{00 \ldots 1}_{n} \ldots \underbrace{99 \ldots \dot{9}}_{n}$$

with the string 99 ... 8 missing, one can apply the identity

$$\frac{10^n(m(10^n-1) + 1)}{(10^n-1)^2} = m + \frac{(m+1)(10^n-1) + 1}{(10^n-1)^2}$$

with $m = 0,1, \ldots, 10^n-2$, in the long division $1/(10^n-1)^2$.

2. Prove that at least one of the numbers

$$\pi + e, \quad \pi e$$

is transcendental.

Thanks to Des MacHale for supplying this problem and its solution.

We use the facts that if x and y are both algebraic numbers then so are $x \pm y$ and xy (see Herstein's Topics in Algebra, page 172), and also $\sqrt{|x|}$. Thus if both $\pi + e$ and πe are algebraic we deduce that

$$\pi = \tfrac{1}{2}(((\pi+e)^2 - 4\pi e)^{\frac{1}{2}} + (\pi+e))$$

is algebraic, which is clearly false.

The argument clearly holds for any pair of transcendental numbers α, β and Des points out that there are generalisations

to more than two numbers, involving the symmetric functions.

3. Suppose that $a_n \geq 0$, for $n = 1,2, \ldots$. How large can

$$\sum_{n=1}^{\infty} \frac{a_n}{e^{a_1+a_2+ \ldots a_n}} \qquad (*)$$

be?

Tom Carroll (a postgraduate at the OU) recently encountered a series of this form while constructing a certain subharmonic function.

In fact the series is convergent with sum less than 1. One can see this by noting that

$$\frac{a_n}{e^{a_1+a_2+ \ldots +a_n}} \leq \frac{e^{a_n} - 1}{e^{a_1+ a_2+ \ldots +a_n}}$$

$$= \frac{1}{e^{a_1+a_2+ \ldots +a_{n-1}}} - \frac{1}{e^{a_1+a_2+ \ldots +a_n}},$$

since $e^x \geq 1+x$. Thus, by telescoping cancellation, the nth partial sum of (*) is at most

$$1 - \frac{1}{e^{a_1+a_2+ \ldots +a_n}} < 1.$$

To see that the number 1 is best possible here, consider

$$\sum_{n=1}^{\infty} \frac{a}{e^{na}} = \frac{a}{e^a - 1}, \qquad a > 0,$$

and notice that

$$\lim_{a \to 0} \frac{a}{e^a - 1} = 1.$$

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IRISH MATHEMATICAL SOCIETY BULLETIN

CONSOLIDATED INDEX OF ARTICLES

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